

A CHARACTERIZATION OF THE 2-FUSION SYSTEM OF $L_4(q)$

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ABSTRACT. We study a saturated fusion system \mathcal{F} on a finite 2-group S having a Baumann component based on a dihedral 2-group. Assuming $\mathcal{F} = O^2(\mathcal{F})$, $O_2(\mathcal{F}) = 1$, and the centralizer of the component is a cyclic 2-group, it is shown that \mathcal{F} is uniquely determined as the 2-fusion system of $L_4(q_1)$ for some $q_1 \equiv 3 \pmod{4}$. This should be viewed as a contribution to a program recently outlined by M. Aschbacher for the classification of simple fusion systems at the prime 2. The corresponding problem in the component-type portion of the classification of finite simple groups (the $L_2(q)$, A_7 standard form problem) was one of the last to be completed, and was ultimately only resolved in an inductive context with heavy artillery. Thanks primarily to requiring the component to be Baumann, our main arguments by contrast require only 2-fusion analysis and transfer. We deduce a companion result in the category of groups.

Saturated fusion systems (or Frobenius categories) are categories, defined by Puig, codifying simultaneously the properties of G -conjugacy of p -subgroups in a finite group, and of Brauer pairs associated to a p -block of a group algebra for G . The study of such p -fusion in finite groups began in the last decade of the 19th century with Burnside and Frobenius. In the latter half of the twentieth, the analysis of fusion was indispensable for the classification of finite simple groups. Abstract fusion theory has evolved in the last decade via the work of many to become the foundation for investigation of spaces which behave like p -completed classifying spaces of finite groups as well as a natural setting for studying the p -local structure of finite groups. In the latter setting, the structure theory of saturated fusion systems parallels that of finite groups. A saturated fusion system has appropriate analogues of $O_p(G)$, $O^p(G)$ and $O^{p'}(G)$ [BCG⁺07], a transfer map [BLO03], normal subgroups and quotients, simplicity, components, layer $E(G)$, and the generalized Fitting subgroup [Asc08, Asc11].

With this groundwork in place, Aschbacher has proposed [AKO11, Section II.13-15] a program for the classification of simple fusion systems at the prime 2, and has begun to carry out substantial parts of it [Asc10, Asc13b, Asc13a, Asc13c]. See also work of Henke [Hen11] and Welz [Wel12]. One reason for doing this is to effect a directed search for new exotic 2-fusion systems other than the Solomon systems. A more central aim is to simplify portions of the classification of finite simple groups. Working in the category of fusion systems provides a clean separation of the analysis of fusion in a finite group from other considerations, such as the obstruction of the existence of cores of local subgroups (which vanishes in the fusion system setting), and of group recognition from local structure. This suggests such simplifications might occur most prominently in the component-type case.

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Early examples of this can be seen in Aschbacher's E -balance theorem [Asc11, Theorem 7] and the Dichotomy Theorem [AKO11, II.14.3] for saturated fusion systems.

Denote by $J(S)$ the Thompson subgroup of S generated by elementary abelian subgroups of S of maximum rank. The Baumann subgroup $\text{Baum}(S)$ is $C_S(\Omega_1(ZJ(S)))$ [AS04, B.2.2]. Let ν_2 be the 2-adic valuation. In this paper, we make a contribution to Aschbacher's program in the (Baumann) component-type case by carrying out a standard form problem for the 2-fusion system of $L_2(q)$ (q odd) resulting in the following characterization of the 2-fusion system of $L_4(q_1)$ ($q_1 \equiv 3 \pmod{4}$).

Theorem A. *Let \mathcal{F} be a saturated fusion system on the 2-group S with $O^2(\mathcal{F}) = \mathcal{F}$ and $O_2(\mathcal{F}) = 1$. Let $x \in S$ be a fully \mathcal{F} -centralized involution, set $\mathcal{C} = C_{\mathcal{F}}(x)$, $T = C_S(x)$, and suppose \mathcal{K} is a perfect normal subsystem of \mathcal{C} . Assume*

- (1) \mathcal{K} is a fusion system on a dihedral group of order 2^k ,
- (2) $Q := C_T(\mathcal{K})$ is cyclic, and
- (3) $\text{Baum}(S) \leq T$.

Then $S \cong D_{2^k} \wr C_2$, and $\mathcal{F} \cong \mathcal{F}_S(G)$ where $G \cong L_4(q_1)$ for some $q_1 \equiv 3 \pmod{4}$ with $\nu_2(q_1 + 1) = k - 1$.

Proof. From Lemma 2.10, S is of 2-rank 3 or 4. Proposition 3.18 says that x does not lie in the center of S , while Theorems 4.1 and 5.1 show that S is of 2-rank 4 and Q is of order at least 4. Finally, Theorem 6.1 identifies \mathcal{F} and completes the proof of the theorem. \square

Here, \mathcal{K} is determined as the unique simple saturated fusion system on a nonabelian dihedral group of the given order, i.e. as $\mathcal{F}_2(L_2(q))$ for some (any) $q \equiv \pm 1 \pmod{8}$ with $\nu_2(q^2 - 1) = k + 1$. Hypothesis (2) means in particular that \mathcal{K} should be regarded as a "standard subsystem" of \mathcal{F} by analogy with [Asc75]. Currently there is no notion of a standard subsystem \mathcal{K} of a fusion system, which presumably would say in particular that $C_{\mathcal{F}}(\mathcal{K})$ is tightly embedded in \mathcal{F} . Aschbacher has defined an appropriate notion of tightly embedded subsystem [Asc13c]. The main obstruction seems to be what one means by the centralizer $C_{\mathcal{F}}(\mathcal{K})$, and more generally by the normalizer $N_{\mathcal{F}}(\mathcal{K})$ of a subsystem. In [Asc11], Aschbacher defines the centralizer of a *normal* subsystem, and we rely on this to talk about $C_T(\mathcal{K})$ as in (2). Hypotheses (1) and (2) are equivalent to specifying the structure of the generalized Fitting subsystem of \mathcal{C} as $F^*(\mathcal{C}) = \mathcal{F}_Q(Q) \times \mathcal{K}$, and \mathcal{K} should be standard subsystem of \mathcal{F} under any appropriate definition. We note Matthew Welz considers the complementary situation of Theorem A, where $C_T(\mathcal{K})$ has 2-rank at least 2, and assuming (1) (but not (3)); see [Wel12].

Let S be an arbitrary finite 2-group, \mathcal{F} a saturated fusion system over S and W a weakly \mathcal{F} -closed subgroup of S . Then \mathcal{F} is of W -characteristic 2-type if $N_{\mathcal{F}}(P)$ is constrained for every fully \mathcal{F} -normalized $1 \neq P \leq S$, and of W -component type if there is a fully \mathcal{F} -centralized involution x with the property that $C_{\mathcal{F}}(x)$ has a component. In the latter case, we say that a subsystem \mathcal{K} is a W -component if it is a component in some $C_{\mathcal{F}}(x)$ with the property that $W \leq C_S(x)$.

Hypothesis (3) is the statement that \mathcal{K} is a $\text{Baum}(S)$ -component and thus \mathcal{F} is of $\text{Baum}(S)$ -component type (alternatively, Baumann component type). The reason one might want to make this restriction in the context of a classification of simple 2-fusion

systems is detailed in [AKO11, §II.14]. As regards our specific situation, (3) allows us to avoid building the Sylow 2-subgroups of certain larger groups in characteristic 2. Without (3) for instance, the fusion systems of $\text{Aut}(L_5(2))$, $\text{Aut}(U_5(2))$, and $\text{Aut}(Sp_4(4))$ appear, as these have involutory automorphisms fixing $L_2(9) \cong \Omega_5(2) \cong Sp_4(2)'$. In the group case, these, HS , and $\text{Aut}(He)$ are identified by Fritz [Fri77] and (independently) by Harris-Solomon [HS77] and Harris [Har77]. All of these almost simple groups are of component type. However, all but $\text{Aut}(He)$ (which has a $2^2L_3(4)$ Baumann component) are of Baumann characteristic 2-type, as are their 2-fusion systems. More seriously, (3) permits us to avoid an inductive approach like that taken by Harris [Har81], where the solution to a large number of other standard form problems were required together with a K -group hypothesis in the midst of a still-lingering unbalanced group conjecture. It was here that the natural targets, $L_4(q^{\frac{1}{2}})$ ($q^{\frac{1}{2}} \equiv 3 \pmod{4}$) and $U_4(q^{\frac{1}{2}})$ ($q^{\frac{1}{2}} \equiv 1 \pmod{4}$) appeared (together with, e.g., $\text{Aut}(\Omega_8^-(q^{\frac{1}{4}}))$ via an appeal to Aschbacher's Classical Involution Theorem. We note that $L_4(q_1)$ and $U_4(q_2)$ have equivalent fusion systems at the prime 2 whenever their Sylow 2-subgroups are isomorphic; this follows from a more general theorem due to Broto, Møller, and Oliver [BMO12, Theorem 3.3].

For the heart of the arguments in this paper, only elementary 2-group analysis, fusion, and transfer are required. Use of transfer is made via the Thompson-Lyons transfer lemma for fusion systems [Lyn13]; see Subsection 1.7. At the beginning of the analysis, we encounter a difficulty unique to the fusion system setting in getting ahold on the structure of subsystems of $\mathcal{C}/C_{\mathcal{C}}(\mathcal{K})$ containing \mathcal{K} , which a priori contain among them exotic extensions of \mathcal{K} . However, work of Andersen, Oliver, and Ventura [AOV12] shows that this is in fact not the case provided certain higher limits associated to \mathcal{K} vanish; see Subsection 1.8. At the end, after determining the structure of S , we apply a piece of Oliver's classification of fusion systems on 2-groups of sectional rank at most 4 [Oli] in order to identify \mathcal{F} .

When combined with a theorem of David Mason [Mas73] and Glauberman's Z^* -theorem, we obtain the following companion of Theorem A in the category of groups. Recall $Z^*(G)$ is the preimage in G of $Z(G/O_{2'}(G))$.

Theorem B. *Let G be a fusion simple finite group (i.e. with $G = O^2(G)$ and $Z^*(G) = O_{2'}(G)$), $S \in \text{Syl}_2(G)$, and $x \in \Omega_1(ZJ(S))$. Assume*

- (1) $C = C_G(x)$ has a perfect normal subgroup K with $K/O_{2'}(K) \cong L_2(q)$ ($q \equiv \pm 1 \pmod{8}$) or A_7 , and
- (2) $C_C(K/O_{2'}(K))$ has cyclic Sylow 2-subgroups.

Then $K/O_{2'}(K) \cong L_2(q)$, q is a square, and $O^{2'}(G/O_{2'}(G)) \cong L_4(q^{\frac{1}{2}})$ if $q^{\frac{1}{2}} \equiv 3 \pmod{4}$, while $O^{2'}(G/O_{2'}(G)) \cong U_4(q^{\frac{1}{2}})$ if $q^{\frac{1}{2}} \equiv 1 \pmod{4}$.

The proof of Theorem B, assuming Theorem A, is found near the beginning of Section 2.

Notation. Homomorphisms are applied on the right. We prefer to write conjugation-like maps in the exponent. For instance, the image of an element $s \in S$ (or subgroup $P \leq S$) under a morphism φ in a fusion system is denoted s^φ (or P^φ).

- C_n is the cyclic group of order n , D_{2^n} ($n \geq 2$), Q_{2^n} ($n \geq 3$), SD_{2^n} ($n \geq 4$) are the dihedral, quaternion, and semidihedral groups, respectively, of order 2^n

- $G^\#$ is the set of nonidentity elements of G
- $\mathcal{I}_p(G)$ is the set of elements of G of order p
- $\Omega_n(P) = \langle x \in P \mid x^{p^n} = 1 \rangle$
- $\mathcal{U}^n(P) = \langle x^{p^n} \mid x \in P \rangle$
- $\mathcal{E}_{p^n}(P)$ is the set of elementary abelian subgroups of P of order p^n
- Nonstandardly, for P with $Z(P)$ of order 2, $P \wr C_2$ is the quotient of $P \wr C_2$ by its center (a wreathed commuting product).

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1. BACKGROUND

We recall in this section (mostly) known results needed for the proof of Theorem A. Our main references for group theoretic material are [Gor80], [Suz82], and [GLS05]. For the background on fusion systems, we follow [AKO11] and [Cra11b].

1.1. Automorphism groups of p -groups. We list some results about automorphism groups of p -groups needed later.

Theorem 1.1. *Let A be a p' -group of automorphisms of the p -group S which stabilizes a normal series $1 = S_0 \leq S_1 \leq \dots \leq S_n = S$ and acts trivially on each factor S_{i+1}/S_i . Then $A = 1$.*

Proof. See for example [Gor80, Theorem 3.2]. □

A finite group is *indecomposable* if it is not the direct product of two proper subgroups.

Proposition 1.2. *An automorphism of a direct product of indecomposable finite groups permutes the commutator subgroups of the factors.*

Proof. This is a consequence of the Krull-Schmidt theorem for finite groups, found in [Suz82, Theorem 2.4.8]. A proof of the current statement is given in [Oli13, Proposition 3.1]. □

Next we describe the outer automorphism group of a nonabelian dihedral 2-group D , and list a couple of additional statements which express the fact that a noncentral involution of D cannot be a commutator or a square in a 2-group containing D as a normal subgroup.

Lemma 1.3. *Let D be a 2-group isomorphic to $D_{2^{k+1}}$ for some $k \geq 2$. Fix the presentation $\langle b, c \mid b^2 = c^{2^k} = 1, b^{-1}cb = c^{-1} \rangle$ for D and let $C = \langle c \rangle$ be the cyclic maximal subgroup of D . Let S be any 2-group containing D as a normal subgroup. Then*

- $\text{Out}(D) \cong A \times B$ where $A \cong C_2$ and $B \cong C_{2^{k-2}}$ is the kernel of the action of $\text{Out}(D)$ on the D -classes of four subgroups of D .
- $[S, S] \leq C_S(C)$, and
- if S_0 is the preimage of $\Omega_1(S/C_S(D)D)$ in S , then $\mathcal{U}^1(S_0) \leq C_S(D)C$.

Proof. For (a), A may be generated by the class $[\eta]$ of the automorphism η sending $b \mapsto c^{-1}b$ and inverting c , and B may be generated by the class $[\varphi]$ of φ centralizing b and sending $c \mapsto c^5$.

For (b), fix a 2-group S containing D as a normal subgroup. Then $C \trianglelefteq S$ as C is a characteristic subgroup of D . Hence (b) follows from the exact sequence $1 \rightarrow C_S(C) \rightarrow S \rightarrow \text{Aut}_S(C) \rightarrow 1$ and the fact that $\text{Aut}(C)$ is abelian.

Let S_0 be the preimage of $\Omega_1(S/C_S(D)D)$ in S as in (c). Thus, S_0 consists of the elements of S which square into $C_S(D)D$. Let $s \in S_0$. If $s \in C_S(D)D$ we have that s squares into $C_S(D)\mathcal{U}^1(D) = C_S(D)\mathcal{U}^1(C)$, so we may assume that s induces a nontrivial (involutory) outer automorphism of D . If s induces some element of the coset $B[\eta]$ as in (a), then s centralizes no noncentral involution of D , and hence $s^2 \in C_S(D)C$ as claimed. So we may further assume that s induces an involutory outer automorphism in B . Then $c^s = cz$ or $c^{-1}z$, and so $s^2 \in C_{C_S(D)D}(C) = C_S(D)C$ as claimed. \square

Lemma 1.4. *Suppose $k \geq 3$ and let D be a 2-group isomorphic to $Q_{2^{k+1}}$, $SD_{2^{k+1}}$, $C_2 \times D_{2^k}$, $D_{2^k} \times D_{2^k}$, or $D_{2^k} \wr C_2$. Then $\text{Aut}(D)$ is a 2-group.*

Proof. This follows from Lemma 1.3 and [Asc00, 23.3] after applying Theorem 1.1 to an appropriate normal series of D . \square

1.2. Fusion systems. Let G be a group. Write $c_g: x \mapsto g^{-1}xg$ for the conjugation homomorphism induced by $g \in G$. For subgroups H and K denote by $\text{Hom}_G(H, K) = \{c_g \mid g^{-1}Hg \leq K\}$ the set of group homomorphisms from H to K induced by conjugation by elements of the group G . Write $\text{Aut}_G(H)$ for $\text{Hom}_G(H, H)$. When $\varphi: H \rightarrow K$ is any isomorphism, we write the induced map from $\text{Aut}(H) \rightarrow \text{Aut}(K)$ as $\alpha \mapsto \alpha^\varphi$.

Fix a prime p and a p -group S . If G is finite and $S \in \text{Syl}_p(G)$, the fusion system of G at p is the category $\mathcal{F}_S(G)$ with objects the subgroups of S , and with morphisms $\text{Hom}_G(P, Q)$ for $P, Q \leq S$.

A *fusion system* is a category \mathcal{F} with objects the subgroups of S and with morphisms injective group homomorphisms between subgroups, containing $\text{Hom}_S(P, Q)$ for each $P, Q \leq S$. In addition, each morphism in a fusion system is assumed to factor as an inclusion followed by an isomorphism.

Fix a fusion system \mathcal{F} on S . When P and Q are subgroups of S which are \mathcal{F} -isomorphic, we also say that P and Q are \mathcal{F} -conjugate and write $P^\mathcal{F}$ for the set of \mathcal{F} -conjugates of P in \mathcal{F} . Elements x and y are \mathcal{F} -conjugate if $\langle x \rangle$ and $\langle y \rangle$ are \mathcal{F} -conjugate by a morphism which sends x to y . Note that $\text{Aut}_P(P) = \text{Inn}(P) \trianglelefteq \text{Aut}_\mathcal{F}(P) \leq \text{Aut}(P)$. Write $\text{Out}_\mathcal{F}(P) = \text{Aut}_\mathcal{F}(P) / \text{Aut}_P(P)$ for the \mathcal{F} -outer automorphism group of P .

Definition 1.5. Let \mathcal{F} be a fusion system on S . A subgroup P of S is said to be

- *fully \mathcal{F} -centralized* if $|C_S(P)| \geq |C_S(Q)|$ for every $Q \in P^\mathcal{F}$,
- *fully \mathcal{F} -normalized* if $|N_S(P)| \geq |N_S(Q)|$ for every $Q \in P^\mathcal{F}$,
- *fully \mathcal{F} -automized* if $\text{Aut}_S(P)$ is a Sylow p -subgroup of $\text{Aut}_\mathcal{F}(P)$.
- *\mathcal{F} -centric* if $C_S(Q) \leq Q$ for every $Q \in P^\mathcal{F}$, and
- *\mathcal{F} -radical* if $O_p(\text{Out}_\mathcal{F}(P)) = 1$.
- *weakly \mathcal{F} -closed* if $P^\varphi = P$ for every $\varphi \in \text{Hom}_\mathcal{F}(P, S)$, and
- *strongly \mathcal{F} -closed* if $R^\varphi \leq P$ for every $R \leq P$ and $\varphi \in \text{Hom}_\mathcal{F}(R, S)$.

We refer to [AKO11, I.2.5] for the definition of a *saturated* fusion system in the form used in this paper. Thus \mathcal{F} is saturated if it satisfies the Sylow and extension axioms. The extension axiom says that given an isomorphism $\varphi \in \text{Hom}_{\mathcal{F}}(P, Q)$ with Q fully \mathcal{F} -centralized, there is a morphism $\tilde{\varphi} \in \text{Hom}_{\mathcal{F}}(N_{\varphi}, S)$, such that $\tilde{\varphi}|_P = \varphi$. Here,

$$N_{\varphi} = \{s \in N_S(P) \mid (c_s)^{\varphi} \in \text{Aut}_S(Q)\}.$$

Note $C_S(P)P \leq N_{\varphi}$ for any such isomorphism φ , so every map $P \rightarrow Q$ with Q fully \mathcal{F} -centralized must extend to $C_S(P)P$; we will apply the extension axiom in this special case quite often. A fusion system of a finite group is saturated [AKO11, Theorem 2.3].

We will often say an element $x \in S$ is fully \mathcal{F} -centralized if $\langle x \rangle$ is fully \mathcal{F} -centralized, especially when x is an involution. Following Aschbacher, we will sometimes write \mathcal{F}^c , \mathcal{F}^r , and \mathcal{F}^f for the set of \mathcal{F} -centric, \mathcal{F} -radical, and fully \mathcal{F} -normalized subgroups of S , respectively. Concatenation in the superscript denotes the intersection of the relevant sets. For example, \mathcal{F}^{cr} is the set of subgroups which are both \mathcal{F} -centric and \mathcal{F} -radical. A saturated fusion system \mathcal{F} is determined by the \mathcal{F} -automorphism groups of the subgroups which lie in \mathcal{F}^{fcr} . This is Alperin's fusion theorem for saturated fusion systems [BLO03, A.10].

The most important weakly closed subgroups for our purposes are the Thompson subgroup $J(S)$ generated by elementary abelian subgroups of maximum rank in S , and the Baumann subgroup $\text{Baum}(S) = C_S(\Omega_1(Z(J(S))))$ [AS04, B.2.2]. Each of these are weakly closed in any fusion system over S .

A *subsystem* of a fusion system \mathcal{F} on S is a fusion system \mathcal{E} on a subgroup of S all of whose morphisms are morphisms in \mathcal{F} . For a subgroup T of S , the normalizer $N_{\mathcal{F}}(T)$ is the fusion subsystem on $N_S(T)$ with morphism sets $\text{Hom}_{N_{\mathcal{F}}(T)}(P, Q)$ consisting of those $\varphi \in \text{Hom}_{\mathcal{F}}(P, Q)$ having an extension $\tilde{\varphi} \in \text{Hom}_{\mathcal{F}}(TP, TQ)$ with $T^{\tilde{\varphi}} = T$. The centralizer $C_{\mathcal{F}}(T)$ is the fusion subsystem on $C_S(T)$ with morphism sets $\text{Hom}_{C_{\mathcal{F}}(T)}(P, Q)$ consisting of those $\varphi \in \text{Hom}_{\mathcal{F}}(P, Q)$ having an extension $\tilde{\varphi} \in \text{Hom}_{\mathcal{F}}(TP, TQ)$ with $\tilde{\varphi}|_T = \text{id}_T$. The subgroup T is *normal* in \mathcal{F} if $\mathcal{F} = N_{\mathcal{F}}(T)$, and *central* if $\mathcal{F} = C_{\mathcal{F}}(T)$. The normalizer (resp. centralizer) is saturated if T is fully \mathcal{F} -normalized (resp. fully \mathcal{F} -centralized). We will often use without comment that $C_{\mathcal{F}_S(G)}(T) = \mathcal{F}_{C_S(T)}(C_G(T))$ in a finite group, as is easily observed (and the same holds for normalizers). Write $O_p(\mathcal{F})$ for the largest normal subgroup of \mathcal{F} , and $Z(\mathcal{F})$ for the *center* of \mathcal{F} , the largest central subgroup.

We need a version of Burnside's fusion theorem, which will often be applied in the case $T = J(S)$.

Lemma 1.6 (Burnside's Fusion Theorem). *Let \mathcal{F} be a saturated fusion system on the p -group S , and suppose that T is a weakly \mathcal{F} -closed subgroup of S . Then any morphism in \mathcal{F} between subgroups of $Z(T)$ lies in $N_{\mathcal{F}}(T)$.*

Proof. Suppose P and Q are subgroups of $Z(T)$, and let $\varphi \in \text{Hom}_{\mathcal{F}}(P, Q)$. Let $\psi \in \text{Iso}_{\mathcal{F}}(Q, Q')$ with Q' fully \mathcal{F} -centralized. By the extension axiom, $\varphi\psi$ and ψ have extensions to $C_S(P)$ and $C_S(Q)$ respectively, and these subgroups both contain T . Restricting these extensions to T and using the fact that T is weakly \mathcal{F} -closed, we get automorphisms $\alpha, \beta \in \text{Aut}_{\mathcal{F}}(T)$ such that $(\alpha\beta^{-1})|_P = \varphi$, which is what was to be shown. \square

1.3. Normal subsystems and quotients. Next we consider notions of normality for fusion subsystems.

Definition 1.7. Fix a saturated fusion system \mathcal{F} on the p -group S and a fusion subsystem \mathcal{E} of \mathcal{F} on the strongly \mathcal{F} -closed subgroup T .

- \mathcal{E} is *weakly normal* if \mathcal{E} is saturated, and whenever $P \leq Q \leq T$, $\varphi \in \text{Hom}_{\mathcal{E}}(P, Q)$, and $\psi \in \text{Hom}_{\mathcal{F}}(Q, S)$, then $\varphi^\psi \in \text{Hom}_{\mathcal{E}}(P^\psi, Q^\psi)$.
- \mathcal{E} is *normal* in \mathcal{F} , written $\mathcal{E} \trianglelefteq \mathcal{F}$, if \mathcal{E} is weakly normal and every element $\alpha \in \text{Aut}_{\mathcal{E}}(T)$ has an extension $\tilde{\alpha} \in \text{Aut}_{\mathcal{F}}(C_S(T)T)$ such that $[C_S(T)T, \alpha] \leq T$.
- \mathcal{F} is *simple* if it contains no normal subsystems other than $\mathcal{F}_1(1)$ and \mathcal{F} .

Craven has shown that the two notions of normality give equivalent definitions of simplicity for saturated fusion systems; see [Cra11a].

Morphisms of saturated fusion systems are defined in [AKO11, II.2]. The quotient of a saturated fusion system by a strongly closed subgroup is defined in [AKO11, II.4,5]. See also [Cra11b, Chapter 5].

Lemma 1.8 ([Asc08, Lemmas 8.9, 8.10]). *Assume that $\theta: \mathcal{F} \rightarrow \mathcal{F}^+$ is a surjective morphism of saturated fusion systems, and S_0 is the kernel of $\theta: S \rightarrow S^+$. Then the map $T \mapsto T\theta$ is a bijection between the strongly \mathcal{F} -closed overgroups of S_0 in S and the set of strongly \mathcal{F}^+ -closed subgroups of S^+ .*

A theorem of Craven gives analogues of the second isomorphism theorem.

Theorem 1.9 ([Cra10, Theorem E]). *Let \mathcal{F} be a saturated fusion system on S , and suppose $T_1 \leq T_2$ are strongly \mathcal{F} -closed subgroups of S . Then $\mathcal{F}/T_2 \cong (\mathcal{F}/T_1)/(T_2/T_1)$.*

1.4. The hyperfocal and residual subsystems. We now look at analogues of $O^p(G)$ and $O^{p'}(G)$ in a saturated fusion system.

Definition 1.10. Let \mathcal{F} be a saturated fusion system on the p -group S .

- (a) The \mathcal{F} -focal subgroup is the subgroup of S defined by

$$\begin{aligned} \text{foc}(\mathcal{F}) &= \langle [s, \varphi] \mid \varphi \in \text{Hom}_{\mathcal{F}}(\langle s \rangle, S) \rangle \\ &= \langle [s, \varphi] \mid s \in P \leq S \text{ and } \varphi \in \text{Aut}_{\mathcal{F}}(P) \rangle. \end{aligned}$$

- (b) The \mathcal{F} -hyperfocal subgroup is the subgroup of S defined by

$$\text{hfp}(\mathcal{F}) = \langle [s, \varphi] \mid s \in P \leq S \text{ and } \varphi \in O^p(\text{Aut}_{\mathcal{F}}(P)) \rangle.$$

Here $[s, \varphi] := s^{-1}s^\varphi$ for $s \in S$ and $\varphi \in \text{Hom}(\langle s \rangle, S)$. The equivalence between the two descriptions of $\text{foc}(\mathcal{F})$ in the definition can be seen via Alperin's fusion theorem.

Lemma 1.11. *Let \mathcal{F} be a saturated fusion system on the p -group S . Then*

- (a) $\text{hfp}(\mathcal{F}) \leq \text{foc}(\mathcal{F})$ and both subgroups are strongly \mathcal{F} -closed,
- (b) for any strongly \mathcal{F} -closed subgroup T of S , the quotient \mathcal{F}/T is the fusion system of the (resp. abelian) p -group S/T if and only if $T \geq \text{hfp}(\mathcal{F})$ (resp. $T \geq \text{foc}(\mathcal{F})$), and
- (c) if $S/\text{foc}(\mathcal{F})$ is cyclic, then $\text{foc}(\mathcal{F}) = \text{hfp}(\mathcal{F})$.

Proof. Part (a) is straightforward. We present a concise proof for (b) due to Craven. By Alperin's fusion theorem, \mathcal{F}/T is the fusion system of the p -group S/T if and only if there are no p' -automorphisms in \mathcal{F}/T of subgroups of S/T . Under the surjective morphism $\mathcal{F} \rightarrow \mathcal{F}/T$, this happens if and only if, for each subgroup P of S , each p' -automorphism α of P , we have $[P, \alpha] \leq T$. Since $O^p(\text{Aut}_{\mathcal{F}}(P))$ is generated by the p' elements in $\text{Aut}_{\mathcal{F}}(P)$, we conclude that \mathcal{F}/T is the fusion system of S/T if and only if $T \geq \text{hnp}(\mathcal{F}) = \langle [P, O^p(\text{Aut}_{\mathcal{F}}(P))] \mid P \leq S \rangle$. A similar argument establishes that in addition, S/T is abelian p -group if and only if $T \geq \text{foc}(\mathcal{F})$.

Now suppose $S/\text{foc}(\mathcal{F})$ is cyclic as in (c). Set $S^+ = S/\text{hnp}(\mathcal{F})$ and $\mathcal{F}^+ = \mathcal{F}/\text{hnp}(\mathcal{F})$. Then part (b) and Craven's (second) isomorphism theorem (Theorem 1.9) imply that $\text{foc}(\mathcal{F})^+ = \text{foc}(\mathcal{F}^+)$, and the latter is just the commutator subgroup $[S^+, S^+]$ because \mathcal{F}^+ is the fusion system of the p -group S^+ . Therefore, the commutator quotient $S^+/[S^+, S^+] = S^+/\text{foc}(\mathcal{F})^+ \cong S/\text{foc}(\mathcal{F})$ is cyclic. It follows that $[S^+, S^+] = 1$, i.e. $\text{foc}(\mathcal{F}) = \text{hnp}(\mathcal{F})$ as claimed. \square

Let \mathcal{F} be a saturated fusion system on the p -group S . A fusion subsystem \mathcal{F}_0 on the subgroup $S_0 \leq S$ has *p -power index in \mathcal{F}* if $S_0 \geq \text{hnp}(\mathcal{F})$, and $\text{Aut}_{\mathcal{F}_0}(P) \geq O^p(\text{Aut}_{\mathcal{F}}(P))$ for every $P \leq S_0$, and *p -prime index in \mathcal{F}* if $S_0 = S$ and $\text{Aut}_{\mathcal{F}_0}(P) \geq O^{p'}(\text{Aut}_{\mathcal{F}}(P))$ for every $P \leq S_0$.

The saturated subsystems of p -power index in \mathcal{F} are in one-to-one correspondence with the overgroups T of $\text{hnp}(\mathcal{F})$ in S [BCG⁺07, Theorem 4.3]. Such a subsystem is normal if and only if T is normal in S [Asc11, Section 7]. Hence there is a unique minimal normal subsystem of p -power index, the *hyperfocal subsystem* $O^p(\mathcal{F})$ based on $\text{hnp}(\mathcal{F})$.

We will require the following theorem of Aschbacher, which allows one to consider the product of a p -group with a normal subsystem. See also [Hen13] for a simplification of Aschbacher's construction and proof of saturation.

Theorem 1.12 ([Asc11, 8.21]). *Let \mathcal{F} be a saturated fusion system on the p -group S and let S_0 and T be strongly \mathcal{F} -closed subgroups of S with $S_0 \leq T$. Suppose \mathcal{F}_0 is a normal subsystem of \mathcal{F} on S_0 . Then there exists a saturated fusion subsystem $\mathcal{F}_0 T$ of \mathcal{F} with the following properties.*

- (a) $\mathcal{F}_0 \leq \mathcal{F}_0 T$,
- (b) $\mathcal{F}_0 T / S_0 \cong \mathcal{F}_{T^+}(T^+)$ where $T^+ = T/S_0$, and
- (c) the map $X \mapsto \mathcal{F}_0 X$ is a bijection between the set of subgroups $X \leq T$ containing S_0 and the set of saturated subsystems of $\mathcal{F}_0 T$ containing \mathcal{F}_0 .

Note the particular case of the preceding theorem: if $\mathcal{F}_0 = O^p(\mathcal{F}_0)$, then as $\text{hnp}(\mathcal{F}_0) \leq \text{hnp}(\mathcal{F}_0 T) \leq S_0$ by (b), we have $\mathcal{F}_0 = O^p(\mathcal{F}_0 T)$.

Broto, Castellana, Grodal, Levi, and Oliver [BCG⁺07, Theorem 5.4] (see also [Pui06, Theorem 6.11]) give a description of the fusion subsystems of index prime to p : they are in one-to-one correspondence with overgroups of a subgroup $\Gamma \leq \text{Aut}_{\mathcal{F}}(S)$ containing $\text{Aut}_S(S)$, and hence there is a unique minimal one $O^{p'}(\mathcal{F})$, the *residual subsystem* of \mathcal{F} . The following corollary to this result will suffice for our purposes.

Proposition 1.13. *Let S be a finite p -group with automorphism group a p -group. Then $\mathcal{F} = O^{p'}(\mathcal{F})$ for every saturated fusion system \mathcal{F} on S .*

The following lists the relationship between the hyperfocal and residual subsystems, surjective morphisms, and direct products we will need. (For the definition of a direct product of two fusion systems, we refer to [AKO11, I.6.5].)

Lemma 1.14. *Suppose that \mathcal{F} is a saturated fusion system on the p -group S .*

- (a) *If $\theta: \mathcal{F} \rightarrow \mathcal{F}^+$ is a surjective morphism of fusion systems, then $\theta(O^p(\mathcal{F})) = O^p(\mathcal{F}^+)$ and $\theta(O^{p'}(\mathcal{F})) = O^{p'}(\mathcal{F}^+)$.*
- (b) *If $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ with each \mathcal{F}_i saturated, then $O^p(\mathcal{F}) = O^p(\mathcal{F}_1) \times O^p(\mathcal{F}_2)$ and $O^{p'}(\mathcal{F}) = O^{p'}(\mathcal{F}_1) \times O^{p'}(\mathcal{F}_2)$.*

Proof. As a surjective morphism of fusion systems is surjective on morphisms, part (a) follows from Alperin's fusion theorem and the fact that $O^p(G)$ and $O^{p'}(G)$ are fully invariant subgroups. Part (b) is [AOV12, Proposition 3.4]. \square

Later in Section 6, we will make use of the following special case of a more general theorem of Oliver.

Theorem 1.15 ([Oli13, Theorem C]). *Let \mathcal{F} be a perfect saturated fusion system on a direct product $D_1 \times D_2$ of two nonabelian dihedral 2-groups of the same order. Then $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ where \mathcal{F}_i is a perfect fusion system on D_i .*

The hypotheses of Oliver's theorem require that $\mathcal{F} = O^{2'}(\mathcal{F})$, but this holds in the case stated above by Lemma 1.4 and Proposition 1.13. Also, the \mathcal{F}_i must be perfect by Lemma 1.14(b).

1.5. Centralizers. To date an appropriate notion of the normalizer or centralizer of an arbitrary saturated fusion subsystem has been elusive. Aschbacher has shown that in the case that a subsystem \mathcal{E} is normal in \mathcal{F} , one can define the centralizer $C_{\mathcal{F}}(\mathcal{E})$ in \mathcal{F} of \mathcal{E} , which enjoys many of the properties one would like. The key result underlying the definition is the following theorem.

Theorem 1.16 ([Asc11, (6.7)]). *Let \mathcal{F} be a saturated fusion system on S and let \mathcal{E} be a normal subsystem on T . Let \mathcal{X} denote the set of subgroups $X \leq C_S(T)$ for which $C_{\mathcal{F}}(X)$ contains \mathcal{E} . Then \mathcal{X} has a unique maximal element, denoted by $C_S(\mathcal{E})$ and called the centralizer in S of \mathcal{E} . Moreover, $C_S(\mathcal{E})$ is strongly \mathcal{F} -closed, and there is a normal subsystem $\mathcal{C}_{\mathcal{F}}(\mathcal{E})$ of \mathcal{F} based on $C_S(\mathcal{E})$.*

Most of the time we will use the characterization of $C_S(\mathcal{E})$ in Theorem 1.16, but for the proof of Theorem B, it is more natural to use Aschbacher's direct construction of $C_S(\mathcal{E})$ as follows.

First we recall some terminology, and some ideas from [Asc08, Section 4]. A saturated fusion system \mathcal{F} over S is *constrained* if $C_S(O_p(\mathcal{F})) \leq O_p(\mathcal{F})$. If \mathcal{F} is constrained, then it has a *model* G [AKO11, III.5.10]. That is, G is a finite group with $S \in \text{Syl}_p(G)$, $O_{p'}(G) = 1$ and $C_G(O_p(G)) \leq O_p(G)$ such that $\mathcal{F} = \mathcal{F}_S(G)$. Any two models for \mathcal{F} are isomorphic by an isomorphism which is the identity on S ; we refer to this as the strong uniqueness of models.

Now let \mathcal{E} a normal subsystem of \mathcal{F} on T . Let $U \in \mathcal{E}^c \cap \mathcal{F}^f$. Then $U \in \mathcal{E}^{fc}$ and $\mathcal{E}(U) := N_{\mathcal{E}}(U)$ is saturated and constrained system on $N_S(U)$. Also $UC_S(U) \in \mathcal{F}^{fc}$,

and so $\mathcal{D}(U) := N_{\mathcal{F}}(UC_S(U))$ is saturated and constrained system on $N_T(U)$. Furthermore $\mathcal{E}(U) \trianglelefteq \mathcal{D}(U)$. Let $G(U)$ be a model for $\mathcal{D}(U)$, and $H(U)$ be the unique normal subgroup of $G(U)$ which is a model for $\mathcal{E}(U)$ [Asc08, Theorem 1]. Then $C_{N_S(U)}(H(U))$ is a well-defined subgroup of S by strong uniqueness of $G(U)$. Aschbacher defines

$$I = \bigcap_{U \in \mathcal{E}^c \cap \mathcal{F}^f} C_{N_S(U)}(H(U))$$

and

$$C_S(\mathcal{E}) = \bigcap_{\varphi \in \text{Aut}_{\mathcal{F}}(TC_S(T))} I^{\varphi}.$$

This is motivated by the fact that $\mathcal{E} = \langle \text{Aut}_{\mathcal{E}}(U)^{\varphi} \mid U \in \mathcal{E}^c \cap \mathcal{F}^f \mid \varphi \in \text{Aut}_{\mathcal{F}}(T) \rangle$ [Asc08, Theorem 3] and that the restriction map $\text{Aut}_{\mathcal{F}}(TC_S(T)) \rightarrow \text{Aut}_{\mathcal{F}}(T)$ is surjective.

The model version of the construction makes clear the situation in a finite group, as in the following.

Lemma 1.17. *Let G be a finite group and N a normal subgroup of G . Let $S \in \text{Syl}_p(G)$, $T = S \cap N$, $\mathcal{F} = \mathcal{F}_S(G)$, and $\mathcal{E} = \mathcal{F}_T(N)$. Let $\mathcal{H}(N) = \{N_N(V) \mid V \in \mathcal{E}^{fc}\}$. Then*

- (a) $C_S(\mathcal{E})$ is the unique largest subgroup of Y of S for which $[H, Y] \leq O_{p'}(H)$ for every $H \in \mathcal{H}(N)$,
- (b) $C_S(N/O_{p'}(N)) \leq C_S(\mathcal{E})$, and
- (c) $C_S(\mathcal{E}) = C_S(N/O_{p'}(N))$ if $\text{Aut}(N/O_{p'}(N))$ contains no element of order p centralizing $H/O_{p'}(H)$ for every $H \in \mathcal{H}(N/O_{p'}(N))$.

Proof. For each $U \in \mathcal{E}^{fc}$, set $H_U = N_N(U)$ for brevity. Let $Y \leq S$ and assume $[H_U, Y] \leq O_{p'}(H_U)$ for each such U . Note then that Y centralizes T because $[T, Y] \leq T \cap O_{p'}(H_T) = 1$. Fix $U \in \mathcal{E}^c \cap \mathcal{F}^f \subseteq \mathcal{E}^{fc}$ and set $G_U = N_G(UC_S(U))$. Then $G(U) := G_U/O_{p'}(G_U)$ is a model for $N_{\mathcal{F}}(UC_S(U))$ with Sylow p -subgroup $N_S(U) \geq Y$. Since $H_U \trianglelefteq G_U$, we have $O_{p'}(H_U) = O_{p'}(G_U) \cap H_U$. Hence $H(U) := H_U/O_{p'}(H_U)$ is a normal subgroup of $G(U)$ while being a model for $N_{\mathcal{E}}(U)$, and so $Y \leq I$ with I as above. But $\text{Aut}_{\mathcal{F}}(TC_S(T))$ acts on \mathcal{E}^{fc} , and so $Y \leq C_S(\mathcal{E})$.

Now set $Y = C_S(\mathcal{E})$. Then Y is strongly \mathcal{F} -closed, and $[H_V, Y] \leq O_{p'}(H_V)$ whenever $V \in (\mathcal{E}^c \cap \mathcal{F}^f)^{N_G(TC_S(T))}$ by reversing the argument in the last paragraph. Let $U \in \mathcal{E}^{fc}$. By a Frattini argument, there is $V \in (\mathcal{E}^c \cap \mathcal{F}^f)^{N_G(TC_S(T))} \cap U^{\mathcal{E}}$. Fix such a V , and let $h \in N$ with $V^h = U$. By Theorem 1.16, there is $g \in C_G(Y)$ and $t \in C_G(V)$ with $h = tg$. Hence $V^g = U$ and $[H_U, Y] = [H_V^g, Y^g] = [H_V, Y]^g \leq O_{p'}(H_V)^g = O_{p'}(H_U)$ as required. This completes the proof of (a).

Let $X = C_S(N/O_{p'}(N))$. Then $[T, X] \leq T \cap O_{p'}(N) = 1$. Also, $[H, X] \leq H \cap O_{p'}(N) \leq O_{p'}(H)$ for any subgroup H of N normalized by X , so (b) is immediate from (a).

For (c), assume that $X < Y$. Then Y/X acts on $N/O_{p'}(N)$ and centralizes (the image of) $H/O_{p'}(H)$ for each $H \in \mathcal{H}(N)$. Since $O_{p'}(N) \cap H \leq O_{p'}(H)$, the same is true for each $\overline{H} \in \mathcal{H}(N/O_{p'}(N))$. Now (c) is clear. \square

We will need a lemma examining in a special case how centralizers behave under quotienting by a strongly closed subgroup. First, the following shows that for the purposes

of computing the centralizer in S of a normal subsystem \mathcal{E} of \mathcal{F} , we may restrict to the subsystem $\mathcal{E}S$ of Theorem 1.12.

Lemma 1.18. *Suppose \mathcal{F} is a saturated fusion system on S and \mathcal{E} is a normal subsystem of \mathcal{F} on T . Let $Q \leq C_S(T)$. Then $C_{\mathcal{F}}(Q) \geq \mathcal{E}$ if and only if $C_{\mathcal{E}S}(Q) \geq \mathcal{E}$.*

Proof. Let $\varphi \in \text{Hom}_{\mathcal{E}}(U, V)$ for subgroups U and V of T . If φ lies in $C_{\mathcal{E}S}(Q)$, then it clearly lies in $C_{\mathcal{F}}(Q)$. Suppose $\varphi \in C_{\mathcal{F}}(Q)$. Then φ extends to a morphism $\tilde{\varphi} \in \text{Hom}_{\mathcal{F}}(QU, QV)$ with $\tilde{\varphi}|_Q = \text{id}_Q$, and it suffices to show that $\tilde{\varphi} \in \mathcal{E}S$. By Alperin's fusion theorem applied in \mathcal{E} , it is enough to show this when $U = V$ and $U \in \mathcal{E}^{fc}$. But then $\tilde{\varphi} \in \text{Aut}_{\mathcal{N}(C_S(U)U)}(U)$ with $\mathcal{N}(C_S(U)U)$ in the sense of [Asc11, Notation 8.4], and $\mathcal{E}S$ is generated by such automorphism groups. \square

Lemma 1.19. *Let \mathcal{F} be a saturated fusion system on the p -group S . Suppose that \mathcal{E} is a normal subsystem of \mathcal{F} on the strongly \mathcal{F} -closed subgroup T of S . Set $Q = C_S(\mathcal{E})$, and assume that $Q \cap T = 1$. Then $C_{S/Q}(\mathcal{E}S/Q) = 1$.*

Proof. Without loss of generality we will assume that $\mathcal{F} = \mathcal{E}S$ by Lemma 1.18. By Theorem 1.16, Q is strongly \mathcal{F} -closed so factoring by Q makes sense. Let $\theta: \mathcal{F} \rightarrow \mathcal{F}/Q$ be the surjective morphism of fusion systems and denote passage to the quotient by bars. Since Q and T are normal in S , we have $[Q, T] \leq Q \cap T = 1$. Also, for every $U, V \leq T$,

$$(1.20) \quad \theta_{U,V}: \text{Hom}_{\mathcal{F}}(U, V) \xrightarrow{\cong} \text{Hom}_{\overline{\mathcal{F}}}(\overline{U}, \overline{V}).$$

is a bijection since $Q \cap T = 1$.

Suppose the proposition is false and let $\overline{Q}_1 = C_{\overline{\mathcal{F}}}(\overline{\mathcal{E}}) \neq 1$, where $Q_1 > Q$ is the preimage of \overline{Q}_1 under θ . By Theorem 1.16, \overline{Q}_1 is strongly $\overline{\mathcal{F}}$ -closed. By Lemma 1.8(a), Q_1 is strongly \mathcal{F} -closed. Since $T \trianglelefteq S$, we have $[Q_1, T] \leq T$. But $[Q_1, T] \leq Q$ as well because $[\overline{Q}_1, \overline{T}] = 1$. Therefore, $[Q_1, T] \leq Q \cap T = 1$.

We will show that $\mathcal{E} \leq C_{\mathcal{F}}(Q_1)$, supplying a contradiction. Let U be a fully \mathcal{E} -normalized, \mathcal{E} -centric subgroup of T and let $\varphi \in \text{Aut}_{\mathcal{E}}(U)$. Since U is fully \mathcal{E} -normalized, we have that $\varphi = \varphi' c_t$ for some $\varphi' \in \text{Aut}_{\mathcal{E}}(U)$ of p' -order and some $t \in T$. If $\varphi' \in C_{\mathcal{F}}(Q_1)$, then so is φ since Q_1 centralizes T , and so we may assume that φ has order prime to p .

Let $\overline{\varphi} = (\varphi)\theta$. Then by definition of \overline{Q}_1 , $\overline{\varphi}$ extends to $\overline{\varphi}_1 \in \text{Aut}_{\overline{\mathcal{F}}}(\overline{Q}_1 \overline{U})$ such that $\overline{\varphi}_1|_{\overline{Q}_1} = \text{id}_{\overline{Q}_1}$. Let φ_1 be a morphism in $\text{Aut}_{\mathcal{F}}(Q_1 U)$ such that $(\varphi_1)\theta = \overline{\varphi}_1$. Then by (1.20), φ_1 restricts to φ on U . Let $\tilde{\varphi}$ be the p' -part of φ_1 . As φ has order prime to p , $\tilde{\varphi}$ still restricts to φ on U . Furthermore, $\varphi_1|_{Q_1}$ stabilizes the series $1 \leq Q \leq Q_1$ and centralizes Q_1/Q , so the same is true for $\tilde{\varphi}$.

Recall that we have assumed as we may that $\mathcal{F} = \mathcal{E}S$. Quotienting now by T and applying (1.20) with the roles of Q and T interchanged, we have that $\text{Aut}_{\mathcal{F}}(Q) \cong \text{Aut}_{\mathcal{E}S/T}(QT/T) \cong \text{Aut}_{S/T}(Q)$ is a p -group, since $\mathcal{E}S/T$ is the fusion system of the p -group S/T by Theorem 1.12(b). It follows that $\tilde{\varphi}|_Q = \text{id}_Q$. By Theorem 1.1, $\tilde{\varphi} = \text{id}_{Q_1}$. We have thus produced for arbitrary $U \in \mathcal{E}^{fc}$ and $\varphi \in \text{Aut}_{\mathcal{E}}(U)$, an extension $\tilde{\varphi} \in \text{Aut}_{\mathcal{F}}(Q_1 U)$ of φ which restricts to the identity on Q_1 . Therefore, $\mathcal{E} \leq C_{\mathcal{F}}(Q_1)$ by Alperin's fusion theorem, contradicting the maximality of Q . \square

1.6. Components and generalized Fitting subsystem. A saturated subsystem \mathcal{E} of \mathcal{F} is *subnormal* if there is a sequence $\mathcal{E} = \mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \cdots \subseteq \mathcal{E}_n = \mathcal{F}$ with \mathcal{E}_i normal in \mathcal{E}_{i+1} for each i . A saturated fusion system \mathcal{E} is *quasisimple* if $\mathcal{E} = O^p(\mathcal{E})$ and $\mathcal{E}/Z(\mathcal{E})$ is simple. \mathcal{E} is a *component* of \mathcal{F} if \mathcal{E} is a subnormal, quasisimple subsystem of \mathcal{F} .

The intersection of two normal subsystems \mathcal{E}_i on T_i need not be normal. However, in [Asc11, Theorem 1], Aschbacher locates a normal subsystem $\mathcal{E} := \mathcal{E}_1 \wedge \mathcal{E}_2 \trianglelefteq \mathcal{F}$ on $T := T_1 \cap T_2$ which is contained in $\mathcal{E}_1 \cap \mathcal{E}_2$. Then \mathcal{E} is the largest subsystem of $\mathcal{E}_1 \cap \mathcal{E}_2$ on T which is normal in \mathcal{E}_1 and \mathcal{E}_2 . This allows one to speak of the normal subsystem *generated by* a collection of subsystems.

Definition 1.21. Let $\text{Comp}(\mathcal{F})$ denote the set of components of \mathcal{F} . The *layer* of \mathcal{F} , denoted $E(\mathcal{F})$, is the normal subsystem of \mathcal{F} generated by $\text{Comp}(\mathcal{F})$. The *generalized Fitting subsystem* is $F^*(\mathcal{F}) = O_p(\mathcal{F})E(\mathcal{F})$.

Theorem 1.22 ([Asc11, Theorem 6]). *The layer $E(\mathcal{F})$ is a central product of the components of \mathcal{F} . Furthermore, $F^*(\mathcal{F})$ is a central product of $O_p(\mathcal{F})$ and $E(\mathcal{F})$, and $C_{\mathcal{F}}(F^*(\mathcal{F})) = Z(F^*(\mathcal{F}))$.*

1.7. Thompson-Lyons transfer lemma. The main technical tool in the proof of Theorem A is the Thompson-Lyons transfer lemma, proved in [Lyn13]. We recall two consequences of this result for easy reference.

Proposition 1.23. *Let \mathcal{F} be a perfect saturated fusion system on a finite 2-group S . Suppose T is a proper normal subgroup of S with S/T cyclic, and let u be an element of least order in $S - T$. Then u has a fully \mathcal{F} -centralized \mathcal{F} -conjugate in T .*

Proposition 1.24. *Let \mathcal{F} be a perfect saturated fusion system on a finite 2-group S . Suppose T is a proper normal subgroup of S with S/T abelian. Let \mathcal{I} be the set of fully \mathcal{F} -centralized involutions in $S - T$, and suppose that the set $\mathcal{I}T = \{vT \mid v \in \mathcal{I}\}$ is linearly independent in $\Omega_1(S/T)$. Then each involution $u \in S - T$ has a fully \mathcal{F} -centralized \mathcal{F} -conjugate in T .*

Since these are used so often, we illustrate how Proposition 1.23 can be applied together with Alperin's fusion theorem to describe the perfect fusion systems on nonabelian 2-groups of maximal class.

Lemma 1.25. *Let \mathcal{F} be a saturated fusion system on the 2-group S with $\mathcal{F} = O^2(\mathcal{F})$. If S is nonabelian of maximal class, then \mathcal{F} is uniquely determined by S up to isomorphism and one of the following holds.*

- (a) $S \cong D_{2^k}$ with $k \geq 3$, and for any odd prime power q with $\nu_2(q^2 - 1) = k + 1$, we have $\mathcal{F} \cong \mathcal{F}_S(G)$ with $G \cong L_2(q)$,
- (b) $S \cong SD_{2^k}$ with $k \geq 4$, and for any odd prime power $q \equiv 3 \pmod{4}$ with $\nu_2(q + 1) = k - 2$, we have $\mathcal{F} \cong \mathcal{F}_S(G)$ with $G \cong L_3(q)$.
- (c) $S \cong Q_{2^k}$ with $k \geq 3$, and for any odd prime power q with $\nu_2(q^2 - 1) = k$, we have $\mathcal{F} \cong \mathcal{F}_S(G)$ with $G \cong SL_2(q)$.

Proof. If P is a subgroup of S , then P is cyclic, dihedral, semidihedral, or quaternion. Hence P has automorphism group a 2-group unless $P \cong C_2 \times C_2$ or Q_8 . Hence, if P is a

proper \mathcal{F} -radical subgroup of S , then $P \cong C_2 \times C_2$ or Q_8 with $\text{Aut}_{\mathcal{F}}(P)$ isomorphic to S_3 or S_4 , respectively.

Let $C = \langle c \rangle$ be the cyclic maximal subgroup of S , and let $Z(S) = \langle z \rangle \leq C$. Let $P \in \mathcal{F}^{cr}$ and suppose that $P \cong C_2 \times C_2$. Let $u \in P - Z(S)$. Then u lies outside the cyclic maximal subgroup T of S . By Proposition 1.23, u is \mathcal{F} -conjugate into T , and therefore \mathcal{F} -conjugate to z . By the above description of the members of \mathcal{F}^{cr} , P is the unique proper \mathcal{F} -centric and \mathcal{F} -radical subgroup containing u . Therefore, u is in fact $\text{Aut}_{\mathcal{F}}(P)$ -conjugate to z by a morphism of order 3, and $\text{Aut}_{\mathcal{F}}(P) \cong S_3$.

Let $P \in \mathcal{F}^{cr}$ with $P \cong Q_8$. Then S is semidihedral or quaternion, and $P = \langle u, z_1 \rangle$ with $z_1 \in P \cap C$. If S is semidihedral, let $T = \Omega_1(S)$, the dihedral maximal subgroup of S . If S is quaternion, let $T = C$. In either case, u is of least order outside T , and the only elements of T of order 4 are z_1 and z_1^{-1} . Hence by Proposition 1.23, u is \mathcal{F} -conjugate to z_1 . As in the previous paragraph, it follows that $\text{Aut}_{\mathcal{F}}(P) \cong S_4$ unless $S = P$, in which case $\text{Aut}_{\mathcal{F}}(P) \cong A_4$.

We have determined the automorphism groups $\text{Aut}_{\mathcal{F}}(P)$ for $P \in \mathcal{F}^{cr}$. Therefore, \mathcal{F} is uniquely determined by Alperin's fusion theorem, and as is described in (a)-(c). \square

1.8. Tameness and p -power extensions. In this subsection we address the problem of determining the structure of extensions of fusion systems, which is resolved via recent work of Andersen, Oliver, and Ventura [AOV12]. Under the conditions of Theorem A, this problem manifests itself in the determination of an involution centralizer from the description of its generalized Fitting subsystem. For suppose given a saturated fusion system on a 2-group and assume \mathcal{C} is an involution centralizer on the subgroup T with $F^*(\mathcal{C}) = O_2(\mathcal{C})\mathcal{E}$ where \mathcal{E} is quasisimple. One is confronted with the possibility that $\overline{\mathcal{C}} = \mathcal{C}/O_2(\mathcal{C})$ is exotic extension of $\overline{\mathcal{E}} = \mathcal{E}/Z(\mathcal{E})$ even when the latter is realizable by a simple group.

Let \mathcal{F} be a saturated fusion system on S and $\mathcal{F}_0 = O^p(\mathcal{F})$ on S_0 . Assume $O_p(\mathcal{F}_0) = 1$. Theorem A of [AOV12] says that if \mathcal{F}_0 is strongly tamely realizable by G_0 (with $O_{p'}(G) = 1$), then \mathcal{F} is realizable as well. Here we indicate how to follow the proof of that theorem to obtain the additional information that G may be chosen so that $\mathcal{F} = \mathcal{F}_S(G)$ and $\text{Inn}(G_0) \leq G \leq \text{Aut}(G_0)$, provided $C_S(\mathcal{F}_0) = 1$.

Roughly, \mathcal{F}_0 is *tame* if \mathcal{F}_0 is realizable by a finite group G_0 such that every outer automorphism of the canonical linking system of G_0 is induced by an outer automorphism of G_0 . An example of a fusion system \mathcal{F}_0 which is tame, realizable by G_0 , but not tamely realizable by G_0 is obtained with $G_0 = A_7$, which is missing the “diagonal automorphism” present on $\mathcal{F}_2(L_2(q))$, ($q \equiv \pm 7 \pmod{16}$). See Proposition 2.6.

The fusion system \mathcal{F}_0 is *strongly tame* if, in addition, certain higher limits of functors associated to G_0 vanish; this is expressed by saying G_0 lies in a certain class $\mathfrak{G}(p)$ of finite groups. We refer to [AKO11, II. § 3.4] for background on linking systems. We also point to [AOV12] for details on tameness and the precise meaning of $\mathfrak{G}(p)$.

Proposition 1.26. *Let \mathcal{F} be a saturated fusion system on the p -group S . Suppose that $\mathcal{F}_0 = O^p(\mathcal{F})$, $C_S(\mathcal{F}_0) = 1$, and that \mathcal{F}_0 is strongly tamely realized by G_0 with $O_{p'}(G_0) = 1$. Then \mathcal{F} is realized by a finite group G such that $\text{Inn}(G_0) \leq G \leq \text{Aut}(G_0)$.*

Proof. A tame fusion system can always be realized by a finite group with no nontrivial normal p' subgroups by [AOV12, Lemma 2.19], and if the fusion system is strongly tame, then the group can be chosen to lie in $\mathfrak{G}(p)$ as well. Since $G_0 \in \mathfrak{G}(p)$, \mathcal{F}_0 has a unique centric linking system \mathcal{L}_0 . In addition, there is a unique centric linking system \mathcal{L} associated to \mathcal{F} by [AOV12, Proposition 2.12(a)].

Next observe that $Z(\mathcal{F}) = Z(\mathcal{F}_0) = 1$, since $C_S(\mathcal{F}_0) = 1$. As $O_{p'}(G_0) = 1$, any central subgroup of G_0 must be a p -subgroup of S central in \mathcal{F}_0 , so it follows that $Z(G_0) = Z(\mathcal{F}_0) = 1$. Now [AOV12, Proposition 1.31(a)] and [AOV12, Proposition 2.16] apply together to give that \mathcal{F} is tamely realized by a finite group G such that $G_0 \trianglelefteq G$. Now $C_S(G_0) = 1$ again follows from $C_S(\mathcal{F}_0) = 1$. Thus G is isomorphic to a subgroup of $\text{Aut}(G_0)$ containing $\text{Inn}(G_0) \cong G_0$. \square

We will apply Proposition 1.26 in the case where $\mathcal{F}_0 \cong \mathcal{F}_2(L_2(q))$ for appropriate q . Hence we need to know

Proposition 1.27. *Let $K \cong L_2(q)$ for $q \equiv \pm 1 \pmod{8}$. Then $K \in \mathfrak{G}(2)$.*

Proof. It is shown in [Oli06, Proposition 7.5] that classical groups in odd characteristic lie in $\mathfrak{G}(2)$. Alternatively, Propositions 4.2 and 4.6(b) of [Oli06] show that any finite simple group of 2-rank at most 3 lies in $\mathfrak{G}(2)$ from general considerations. \square

2. STRUCTURE OF THE INVOLUTION CENTRALIZER

In this section we lay the groundwork for the study of \mathcal{F} as in Theorem A by studying some consequences of Proposition 1.26, and fixing notation.

First we record some information about $\text{Aut}(L_2(q))$.

Lemma 2.1. *Let K be a finite group isomorphic to $L_2(q)$ and $P \in \text{Syl}_2(K)$. Let $H = \text{Aut}(K)$, and $T \in \text{Syl}_2(H)$. Then*

- (a) $H = K\langle h \rangle F$ with $h \in \mathcal{I}_2(T)$, F is cyclic, and $\text{Out}(K) \cong \langle h \rangle \times F$,
- (b) P is dihedral of order 2^k where $\nu_2(q^2 - 1) = k + 1$,
- (c) $K\langle h \rangle \cong \text{PGL}_2(q)$ and $P\langle h \rangle$ is dihedral,
- (d) F is isomorphic to the Galois group of $GF(q)$ inducing field automorphisms on K ,
- (e) if $F_T := T \cap F$, then $P\langle h \rangle \cap F_T = 1$, and $|F_T| \leq 2^{k-2}$,
- (f) all involutions of P are K -conjugate as are all involutions of Ph ,
- (g) if f is an involution in $F_T := T \cap F$, then $P\langle f \rangle = P \times \langle f \rangle$, q is a square, $C_K(f) \cong \text{PGL}_2(q^{1/2})$, and $P \cap O^2(C_K(f))$ is dihedral of order 2^{k-1} .
- (h) if f is an involution in $F_T := T \cap F$ and $\langle z \rangle = Z(P)$, then $[f, h] = (fh)^2 = z$, $P\langle fh \rangle$ is semidihedral, and all involutions of T lie in $P \cup Ph \cup Pf$.

Proof. This is taken from Chapter 10, Lemma 1.2 of [GLS05]. \square

We are now in a position to prove Theorem B assuming Theorem A.

Proof of Theorem B. Let $\mathcal{F} = \mathcal{F}_S(G)$. We may assume $\langle x \rangle$ is fully \mathcal{F} -centralized by choosing S appropriately. Then $T := C_S(x)$ is a Sylow 2-subgroup of $C = C_G(x)$. Set $P = T \cap K \in \text{Syl}_2(K)$, a dihedral group of order 2^k with $k \geq 3$, and set $\mathcal{K} = \mathcal{F}_P(K)$. Put

$Q = C_T(\mathcal{K})$, $\overline{K} = K/O_{2'}(K)$ and $Q_0 = C_T(K/O(K))$. Then Q_0 is a Sylow 2-subgroup of $C_C(K/O(K))$, which is cyclic by hypothesis.

We check the hypotheses of Theorem A for \mathcal{F} . First we verify that

$$(2.2) \quad Q = Q_0 \text{ is cyclic.}$$

By Lemma 1.17(b,c), it suffices to show that no involution in $H := \text{Aut}(\overline{K})$ centralizes mod core the normalizer of \overline{P} and of each four subgroup of \overline{P} . Assume otherwise that $w \in I_2(\text{Aut}(\overline{K}))$ is such an involution. Also suppose $K = \overline{K}$ to lighten notation. Since w centralizes $N_K(P)$ mod core, it centralizes P . If $w \in Z(P)$, then each four group $V \leq P$ contains w and $N_K(V)$ permutes transitively the involutions in V . So we may assume w induces a nontrivial field automorphism of K by Lemma 2.1(h,e,c) in case $K \cong L_2(q)$, or w is a transposition or a product of three transpositions in case $K \cong A_7$. Suppose $K \cong L_2(q)$. If V is a four subgroup of $C_K(w)$, then Lemma 2.1(g) implies that $N_{C_K(w)}(V)$ permutes transitively the three involutions of $Vw - \{w\}$. Hence zw does not centralize $N_K(V)$ mod core. So $w = (zw)^h$ does not centralize $N_K(V^h)$ mod core, and this is the final contradiction. The argument in the case $K \cong A_7$ is as straightforward. Hence, (2.2) implies (2) of Theorem A is satisfied.

Now we show:

$$(2.3) \quad Q = O_2(\mathcal{C}).$$

By (2.2), $C_C(\mathcal{K})$ is the fusion system of the 2-group Q . By [Asc11, 9.8.2], $\mathcal{K} = E(\mathcal{C})$. Now [Asc11, 9.12] applies to give that $O_2(\mathcal{C}) = F^*(C_C(\mathcal{K})) = Q$, as desired.

Next, we show

$$(2.4) \quad O_2(\mathcal{F}) = 1.$$

Let $Y = O_2(\mathcal{F})$ and suppose that $Y \neq 1$. Then $\Omega_1(Z(Y)) \trianglelefteq \mathcal{F}$. Hence $\Omega_1(Z(Y)) \cap T$ is (nontrivial) normal in $\mathcal{C} = \mathcal{C}_{\mathcal{F}}(x)$; this follows for instance from the Aschbacher-Stancu characterization of normality [Cra11b, Proposition 4.62, Theorem 5.29]. So $\Omega_1(Z(Y)) \cap T \leq \Omega_1(Q) = \langle x \rangle$. If $T < S$, this is a contradiction. So $S = T$ and $\Omega_1(Z(Y)) = \langle x \rangle$ is strongly \mathcal{F} -closed. This contradicts the Z^* -theorem and hence $Y = 1$.

As $\mathcal{F} = O^2(\mathcal{F})$ by Puig's hyperfocal subgroup theorem [Pui00], and $\text{Baum}(S) \leq C_S(x) = T$ by assumption on x , \mathcal{F} satisfies the hypotheses of Theorem A. Hence $S \cong D_{2^k} \wr C_2$ and $\mathcal{F} \cong \mathcal{F}_2(L_4(q_1))$ for some $q_1 \equiv 3 \pmod{4}$. Now a Theorem of David Mason [Mas73, Theorems 1.1, 3.14] shows that $O_2'(G/O_{2'}(G))$ is isomorphic with $L_4(q_2)$ for some $q_2 \equiv 3 \pmod{4}$ or with $U_4(q_3)$ for some $q_3 \equiv 1 \pmod{4}$. From the structure of involution centralizers in $L_4(q_2)$ and $U_4(q_3)$ (see [Suz86, 6.5.2, 6.5.15]) it follows that $q_2^2 = q$ and $q_3^2 = q$ in the respective cases. This completes the proof of Theorem B. \square

We now begin work on the proof of Theorem A. The following hypothesis simply extracts those of Theorem A and fixes some notation.

Hypothesis 2.5. Suppose that \mathcal{F} is a saturated fusion system on the 2-group S with $\mathcal{F} = O^2(\mathcal{F})$ and $O_2(\mathcal{F}) = 1$. Assume x is a fully \mathcal{F} -centralized involution in S with $\text{Baum}(S) \leq C_S(x)$. Write $T = C_S(x)$ and $\mathcal{C} = C_{\mathcal{F}}(x)$, and assume that $\mathcal{K} = E(\mathcal{C})$ is a fusion system on a dihedral group P of order 2^k . Assume $Q = C_T(\mathcal{K})$ is cyclic. Set $R = QP = Q \times P$.

Unless otherwise specified, we assume for the remainder of this paper that \mathcal{F} is a fusion system satisfying Hypothesis 2.5, adopting the notation there. The next Proposition allows us to choose a suitable realization of \mathcal{K} .

Proposition 2.6. *Let P be a nonabelian dihedral group of order 2^k . There is a unique perfect saturated fusion system \mathcal{K} on P . Fix a prime $p \equiv 5 \pmod{8}$, let $q = p^{2^{k-2}}$, and set $K = L_2(q)$. Then \mathcal{K} is strongly tamely realized by K .*

Proof. Lemma 1.25(a) shows there is a unique fusion system \mathcal{K} on P with $\mathcal{K} = O^2(\mathcal{K})$. That \mathcal{K} is tamely realized by K is the content of [AOV12, Proposition 4.3]. Strong tameness follows from Proposition 1.27. \square

For the remainder, we fix an odd prime power $q \equiv \pm 1 \pmod{8}$ with $\nu_2(q^2 - 1) = k + 1$ and K tamely realizing \mathcal{K} as in Proposition 2.6.

Since $\mathcal{K} = E(\mathcal{C})$ is a normal subsystem of \mathcal{C} , for each $T_1 \leq T$ with $P \leq T_1$ we may form the product $\mathcal{K}T_1$ as in Theorem 1.12. Then $O^2(\mathcal{K}T_1) = O^2(\mathcal{K}) = \mathcal{K}$.

Proposition 2.7. *For each $T_1 \leq S$ with $R \leq T_1$, the quotient $\mathcal{K}T_1/Q$ is isomorphic to the 2-fusion system of a subgroup of $\text{Aut}(K)$ containing K .*

Proof. We verify the hypotheses of Proposition 1.26 with $\mathcal{F} = \mathcal{K}T_1/Q$ and $\mathcal{F}_0 = \mathcal{K}Q/Q \cong \mathcal{K}$. Denote quotients by Q with bars. By Proposition 2.6, \mathcal{K} is strongly tamely realized by K and $O_2(K) = 1$. By Lemma 1.14, we have that $\overline{\mathcal{K}} = O^2(\overline{\mathcal{K}T_1})$. Since $Q \cap P = 1$, $C_{\overline{T_1}}(\overline{\mathcal{K}T_1}) = 1$ by Lemma 1.19. Therefore $\mathcal{K}T_1/Q$ is the 2-fusion system of a subgroup of $\text{Aut}(K)$ containing $\text{Inn}(K) \cong K$. \square

Lemma 2.8. *Let $f \in T$ be an involution. If $f \in C_T(P)$ but $f \notin R$, then*

- (a) $\mathcal{K}P\langle f \rangle$ is the fusion system of $K\langle f \rangle$ where f is an involutory field automorphism of K , and
- (b) if V is a four subgroup of P , there exists a unique $i \in \{0, 1\}$ such that $C_{\mathcal{C}}(fz^i)$ contains $\text{Aut}_{\mathcal{K}}(V)$. In this case, if V' is another four subgroup of P not P -conjugate to V , then $C_{\mathcal{C}}(fz^{1-i})$ contains $\text{Aut}_{\mathcal{K}}(V')$.

Proof. By Theorem 1.12(c) the set of saturated subsystems of $\mathcal{K}T$ is in one-to-one correspondence with the set of subgroups of T containing P via the bijection $X \mapsto \mathcal{K}X$. Factoring by Q induces an isomorphism of $\mathcal{K}P\langle f \rangle$ with the fusion system of an extension of K (by Proposition 2.7) containing an involution outside K centralizing a Sylow 2-subgroup of K . This is unique and the required extension by Lemma 2.1, proving (a).

Now by Lemma 2.1(g), $C_{\mathcal{K}P\langle f \rangle}(f)$ contains $\text{Aut}_{\mathcal{K}}(U)$ for each U in some unique P -class of four-subgroups in P . Furthermore, there is an (abstract) $\mathcal{K}P\langle f \rangle$ -fusion preserving isomorphism c_h of $P\langle f \rangle$ which swaps the P -classes of four-subgroups of P and interchanges f and fz by Lemma 2.1(c,h). So either $\text{Aut}_{\mathcal{K}}(V)$ or $\text{Aut}_{\mathcal{K}}(V')$ is contained in $C_{\mathcal{K}P\langle f \rangle}(f)$, and the other \mathcal{K} -automorphism group is contained in $C_{\mathcal{K}P\langle f \rangle}(fz)$. \square

We write $\mathcal{K}\langle f \rangle$ in place of $\mathcal{K}P\langle f \rangle$, for brevity. In view of Lemma 2.8, we also make the following definition in the situation of Hypothesis 2.5.

Definition 2.9. We say that an involution $f \in T$ is an f -element on \mathcal{K} if $f \in C_T(P)$ but $f \notin R$.

Viewing $\overline{T} = T/Q$ as a subgroup of the Sylow 2-subgroup of $\text{Aut}(K)$, let \overline{F} denote the intersection of \overline{T} with group of field automorphisms of $\text{Aut}(K)$, and let F be the preimage in T of \overline{F} . Also, let F_1 be the preimage in T of $\Omega_1(\overline{F})$. Hence, $F_1 \cap R = Q$, $|F_1 : Q| = 1$ or 2 , and $[F_1, P] \leq Q \cap P = 1$ by Lemma 2.1(g). So $C_T(P) = F_1 Z(P)$ by Lemma 2.1(c).

Let R_d denote the largest subgroup of T containing R for which R_d/Q is dihedral of order 2^{k+1} . Thus R_d contains R with index 1 or 2. An argument as in part (a) of Lemma 2.8 applies to give that $\mathcal{K}R_d/Q$ is the 2-fusion system of $L_2(q)$ or $PGL_2(q)$, respectively.

Lemma 2.10. *Assume Hypothesis 2.5. If T contains an f -element on \mathcal{K} , then T (and hence S) has 2-rank 4 and $J(S) = J(RF_1)$. Otherwise T (and hence S) has 2-rank 3.*

Proof. As remarked previously, $C_T(P) = F_1 Z(P)$. Thus, there is an f -element on \mathcal{K} in T if and only if F_1 splits over Q . If $h_1 \in \mathcal{I}_2(T - RF)$, then $C_R(h_1)$ is of 2-rank 2 by Lemma 2.1(c). Lemma 2.1(h) shows that h_1 centralizes no involution in $RF - R$. Hence, $C_T(h_1)$ is of 2-rank 3. Suppose f is an f -element of T . Then as $f \in C_T(P)$ the 2-rank of RF is 4. Hence, $J(T) = J(RF) = J(RF_1)$ in this case, and T is of 2-rank 4. If T contains no f -element, then $J(RF) = J(R)$ is of 2-rank 3, and so T is of 2-rank 3 as well. Since $J(S) \leq \text{Baum}(S) \leq T$, we have $J(S) = J(T)$, and the 2-rank of S is the 2-rank of T . \square

3. THE 2-CENTRAL CASE

We begin now the heart of the analysis of a fusion system \mathcal{F} satisfying Hypothesis 2.5. The objective of the current section is to consider the case in which x lies in the center of S , i.e. in which $S = T$. Eventually, in Proposition 3.18, we will reach the conclusion that there is no such \mathcal{F} . This section and the next are modeled on the treatment in [GLS05], in particular Proposition 3.4 of Chapter 2 and Section 12 of Chapter 3 there.

Adopt the notation of Section 2 and in particular of Hypothesis 2.5. Thus $\mathcal{C} = C_{\mathcal{F}}(x)$, \mathcal{K} is a component of \mathcal{C} on the dihedral group P , $T = C_S(x)$ and $Q = C_T(\mathcal{K})$. Recall the definition of f -element from Definition 2.9 and the definitions of R_d and F . Also set $Z(P) = \langle z \rangle$, $T_0 = \Omega_1(T)$, and $Z = \Omega_1(Z(T_0))$. Directly from the definition of the centralizer and the fact that \mathcal{K} has a single class of involutions, we have

$$(3.1) \quad \text{all involutions of } P^\#x \text{ are } \mathcal{C}\text{-conjugate}$$

We begin with two lemmas which apply throughout this section, after which we state the main technical result of the present case.

Lemma 3.2. *z is weakly \mathcal{F} -closed in Z .*

Proof. As $T_0 = \Omega_1(T)$ is weakly \mathcal{F} -closed in T , $\text{Aut}_{\mathcal{F}}(T_0)$ controls fusion in $Z = \Omega_1 Z(T_0)$ by Lemma 1.6. Suppose z is not weakly \mathcal{F} -closed in Z , and let $\varphi \in \text{Aut}_{\mathcal{F}}(T_0)$ such that $z \neq z^\varphi \in Z$. Since $z \in Z(T_0)$, we may take φ to be of odd order. Set $P_i = P^{\varphi^i}$ for $i \in \mathbf{Z}_{\geq 0}$. As $P \leq T$ and $P_i = \Omega_1(P_i)$ for all i , we have $P_i \leq T_0$ for all i . Then

$$(3.3) \quad Z(P_i) \cap Z(P_j) = 1 \text{ for each } i \neq j \in \{0, 1, 2\}$$

because φ has odd order and $z^\varphi \neq z$. Furthermore, $Z(P_i) \leq Z$ for all i .

Let $i \neq j \in \{0, 1, 2\}$, and suppose that $P_i \cap P_j \neq 1$. As $P_i \cap P_j \leq P_j$, $Z(P_j) \leq P_i \cap P_j$. Hence $Z(P_j) \leq P_i \cap Z = Z(P_i)$, contrary to (3.3). As $[P_i, P_j] \leq P_i \cap P_j$, we have $P_0 P_1 P_2 \cong P_0 \times P_1 \times P_2$. But T has 2-rank at most 4 by Lemma 2.10, so this is a contradiction. \square

Lemma 3.4. *If $\varphi \in \text{Hom}_{\mathcal{F}}(P, T)$, then $z^\varphi = z$.*

Proof. First recall that $P \trianglelefteq T$. Let C be the cyclic maximal subgroup of P . Then $C \trianglelefteq T$. Now $[T, T] \leq R$ from Proposition 2.7, and in fact $[T, T] \leq C_R(C) = QC$ by Lemma 1.3(b). So T/QC is abelian. Thus

$$(3.5) \quad \Omega_1([T, T]) \leq \Omega_1(QC) = \langle x, z \rangle \leq Z.$$

Let $\varphi \in \text{Hom}_{\mathcal{F}}(P, T)$. Then $[u_1, u_2] = z$ for a pair of involutions u_1 and u_2 of P , and so $z^\varphi = [u_1^\varphi, u_2^\varphi] \leq \Omega_1([T, T]) \leq Z$ by (3.5). Now Lemma 3.2 shows that $z^\varphi = z$. \square

Proposition 3.6. *Suppose \mathcal{F} satisfies Hypothesis 2.5 with $S = T$. Then no involution of T is an f -element.*

Assume the hypotheses and notation of the proposition, but that the statement is false. To that end, let f be an involutory f -element in T . We proceed in a series of lemmas.

Lemma 3.7. $z^{\mathcal{F}} \cap R\langle f \rangle \subseteq P$.

Proof. We first show that

$$(3.8) \quad z^{\mathcal{F}} \cap C_T(P) = \langle z \rangle.$$

Let $y \in z^{\mathcal{F}} \cap C_T(P)$ and choose $\varphi \in \mathcal{F}$ with $y^\varphi = z$. Since $z \in Z(T)$, φ extends to a morphism on $P \leq C_T(y)$. Therefore $y = z$ by Lemma 3.4, and (3.8) holds.

Now we suppose the lemma fails and let $y \in z^{\mathcal{F}} \cap (R\langle f \rangle - P \cup C_T(P))$ be arbitrary. We claim that

$$(3.9) \quad y \text{ is } \mathcal{C}\text{-conjugate to an element of } C_T(P).$$

Together with (3.8) and the fact that P is strongly \mathcal{C} -closed, this will yield a contradiction. Recalling that $R\langle f \rangle = Q\langle f \rangle \times P$, write $y = uf_0v$ with $u \in Q$, $f_0 \in \langle f \rangle$, and $v \in P$. Since y is an involution outside $P \cup C_T(P)$, we have $uf_0 \neq 1 = (uf_0)^2$, $v \notin Z(P)$, and $v^2 = 1$. Let V be the four subgroup of P containing v . Then $\text{Aut}_{\mathcal{K}}(V) \leq C_{\mathcal{K}\langle f \rangle}(f_0z^i)$ for some $i \in \{0, 1\}$ by Lemma 2.8, and so there exists a morphism $\varphi \in \text{Aut}_{\mathcal{K}\langle f \rangle}(V\langle f \rangle)$ with $(f_0v)^\varphi = (f_0z^i)^\varphi(z^iv)^\varphi = f_0z^i \cdot z \in \langle f, z \rangle$. As $Q \trianglelefteq \mathcal{C}$, this φ extends to a morphism $\tilde{\varphi} \in \mathcal{C}$ fixing Q , and hence $y^{\tilde{\varphi}} = (uf_0v)^{\tilde{\varphi}} \in Q\langle f, z \rangle = C_T(P)$ confirming (3.9). \square

Lemma 3.10. P is weakly \mathcal{F} -closed.

Proof. Suppose not. Choose by Alperin's fusion theorem a fully \mathcal{F} -normalized subgroup $D \leq T$ containing P and an automorphism $\varphi \in \text{Aut}_{\mathcal{F}}(D)$ with $P^\varphi \neq P$. Since P is normal in T , we can choose such a φ of odd order. Set $P_i = P^{\varphi^i}$ for each i . Thus the subgroups $P_0 = P$, P_1 and P_2 are distinct by choice of φ , whereas $Z(P_i) = Z(P_0) = \langle z \rangle$ for all i by Lemma 3.4.

Now we examine the images of the P_i in $\overline{T} = T/Q$. Since $\overline{z} \neq 1$, we have that $\overline{P_i} \cong D_{2^k}$ for all i . Furthermore, $P_i = \Omega_1(P_i) \leq \langle z^{\mathcal{F}} \rangle$ as all involutions of P are \mathcal{F} -conjugate. Thus there exists $h \in R_d - R$ squaring into Q , and

$$\overline{P_i} \leq \overline{P}\langle \overline{h} \rangle$$

for all i by Lemma 3.7 and the fact (Lemma 2.1(h)) that there are no involutions in $\overline{P}hf$.

Suppose that $\overline{P} = \overline{P}_i$ for some i . Then $PQ \geq P_i$, and so $\mathcal{I}_2(P_i) \subseteq R$. By Lemma 3.7 then, $P_i \leq \langle z^{\mathcal{F}} \cap P_i \rangle \leq P$. So $P = P_i$. This shows that $\overline{P}_0 \neq \overline{P}_1$ and $\overline{P}_0 \neq \overline{P}_2$. But $\overline{P}\langle \overline{h} \rangle$ is dihedral and the \overline{P}_i are among the two dihedral maximal subgroups of $\overline{P}\langle \overline{h} \rangle$ so

$$(3.11) \quad \overline{P}_1 = \overline{P}_2.$$

Set $S_0 = P_0P_1$ and $S_1 = P_1P_2$, so that $S_0^{\varphi} = S_1$. Then $\overline{S}_0 = \overline{P}\langle \overline{h} \rangle \cong D_{2^{k+1}}$ but

$$(3.12) \quad \overline{S}_1 \cong D_{2^k}$$

from (3.11).

As $\overline{P}_0 \neq \overline{P}_1$ are dihedral maximal subgroups of $\overline{P}\langle \overline{h} \rangle$, we have $[\overline{P}_0, \overline{P}_1]$ is the cyclic maximal subgroup of \overline{P}_0 . So $[P_0, P_1]$ is the cyclic maximal subgroup of P_0 . But $[P_0, P_1] \leq P_1$ because the P_i normalize each other, and hence $[P_0, P_1]$ is the cyclic maximal subgroup of P_1 as well. It follows that $P_0 \cap P_1$ has index 2 in P_0 and P_1 , and $|S_0| = 2|P_0| = 2^{k+1}$. Hence, $S_0 \cong \overline{S}_0 \cong D_{2^{k+1}}$ and $S_1 = S_0^{\varphi}$ is also isomorphic to $D_{2^{k+1}}$ with center $\langle z \rangle^{\varphi} = \langle z \rangle$. As $\overline{z} \neq 1$, $\overline{S}_1 \cong D_{2^{k+1}}$. This contradicts (3.12) and completes the proof. \square

Lemma 3.13. *Let u be an involution in $C_T(P)$. If u is fully \mathcal{F} -centralized, then so is uz .*

Proof. Let $\varphi \in \text{Hom}_{\mathcal{F}}(\langle uz \rangle, T)$ with $(uz)^{\varphi}$ fully \mathcal{F} -centralized. Then φ extends to a morphism $\tilde{\varphi}$ on $C_T(uz) = C_T(u) \geq P$, and $z^{\tilde{\varphi}} = z$ by Lemma 3.4. Since u is fully \mathcal{F} -centralized, we have

$$|C_T(uz)| = |C_T(u)| \geq |C_T(u^{\tilde{\varphi}})| = |C_T(u^{\tilde{\varphi}}z)| = |C_T((uz)^{\tilde{\varphi}})|$$

and so uz is fully \mathcal{F} -centralized as well. \square

Lemma 3.14. *There exists $f_0 \in f\langle z \rangle$ and $\varphi \in N_{\mathcal{F}}(P)$ such that $f_0^{\varphi} = x$.*

Proof. We have $R \trianglelefteq T$ with T/R abelian. By Lemma 2.1(h), all involutions in $T - R$ lie in $Rh \cup Rf$. As $\mathcal{F} = O^2(\mathcal{F})$, there exists $\varphi \in \text{Hom}_{\mathcal{F}}(\langle f \rangle, R)$ with f^{φ} fully \mathcal{F} -centralized by Proposition 1.24. Then φ extends (by the extension axiom) to a morphism $\tilde{\varphi}$ on $C_T(f) \geq P$ normalizing P by Lemma 3.10. Thus, $\tilde{\varphi} \in N_{\mathcal{F}}(P)$ and so $f^{\tilde{\varphi}} \in \Omega_1(C_R(P)) - \langle z \rangle = x\langle z \rangle$ as $[P, f] = 1$. Since one of $f^{\tilde{\varphi}}$ or $(fz)^{\tilde{\varphi}} = f^{\tilde{\varphi}}z$ equals x , we are finished. \square

Now the next two lemmas give a contradiction in the proof of Proposition 3.6.

Lemma 3.15. $xz \in x^{\mathcal{F}}$.

Proof. Suppose not. Since x is not \mathcal{F} -conjugate to z by Lemma 3.7, and all involutions of $P^{\#}x$ are \mathcal{F} -conjugate by Lemma 3.1, we have that

$$(3.16) \quad \langle x \rangle \text{ is weakly } \mathcal{F}\text{-closed in } \langle x \rangle \times P.$$

Replacing f by fz if necessary, there exists a subgroup $D \leq T$ with $P \leq D$ and a morphism $\varphi \in \text{Hom}_{\mathcal{F}}(D, T)$ with $x^{\varphi} = f$ and $P^{\varphi} = P$ by Lemma 3.14. By Lemma 2.8(b), $\langle f \rangle$ is not weakly closed in $\mathcal{K}\langle f \rangle$ and so there exists $\psi \in \text{Hom}_{\mathcal{F}}(\langle f \rangle, \langle f \rangle \times P)$ with $f^{\psi} \in Pf - \{f\}$. Then $x^{\varphi\psi\varphi^{-1}} \in Px - \{x\}$, which contradicts (3.16) and completes the proof. \square

Lemma 3.17. $xz \notin x^{\mathcal{F}}$.

Proof. Let $\varphi \in \mathcal{F}$ with $x^\varphi = xz$. Then as x and z lie in $Z(T)$ we may assume $\varphi \in \text{Aut}_{\mathcal{F}}(T)$ is of odd order by the extension and Sylow axioms. Then $z^\varphi = z$ by Lemma 3.10, and so φ induces an automorphism of $\langle x, z \rangle$ of order 2, a contradiction. \square

This completes the proof of Proposition 3.6. We now can prove the main result of this section.

Proposition 3.18. *Suppose \mathcal{F} satisfies Hypothesis 2.5. Then $T < S$.*

Proof. Suppose to the contrary that $T = S$. From Proposition 3.6, there is no involutory f -element in T , and hence there are no involutions in Rf . By Lemma 2.1(h), it follows that

$$(3.19) \quad T_0 \leq R_d$$

and hence $Z = \langle x, z \rangle \leq Z(T)$ in the present case. Furthermore,

$$(3.20) \quad T \text{ is of 2-rank 3}$$

by Lemma 2.10.

In view of Lemma 3.2, we know z is not \mathcal{F} -conjugate to x or xz . Since fusion in $Z(T)$ is controlled in $\text{Aut}_{\mathcal{F}}(T)$,

$$(3.21) \quad x, xz, \text{ and } z \text{ are pairwise not } \mathcal{F}\text{-conjugate.}$$

Our assumption that $O_2(\mathcal{F}) = 1$ yields that x has an \mathcal{F} -conjugate outside Z . Apply Alperin's fusion theorem to obtain a fully \mathcal{F} -normalized, \mathcal{F} -centric subgroup D of T , and an automorphism $\alpha \in \text{Aut}_{\mathcal{F}}(D)$ with $x^\alpha \notin Z$. Set $h = x^\alpha$.

Note that $h \notin R$, as otherwise h would lie in $\Omega_1(R) = \langle x \rangle \times P$. Because all involutions of $P \cup P^\#x$ are \mathcal{F} -conjugate to xz or z , (3.21) would yield $h = x$, contrary to the choice of h . Therefore, by (3.19):

$$(3.22) \quad h \in R_d - R \text{ and } P\langle h \rangle \text{ is dihedral.}$$

Since D is \mathcal{F} -centric, it contains $Z = \langle x, z \rangle \leq \Omega_1(Z(T))$, and hence $\Omega_1(Z(D)) = Z\langle h \rangle$ as T is of 2-rank 3. Set $A = Z\langle h \rangle$. Then h is $N_P(A)$ -conjugate to hz by (3.22). If h is $\text{Aut}_{\mathcal{F}}(A)$ -conjugate to hx or hxz then it is $\text{Aut}_{\mathcal{F}}(A)$ -conjugate to both, so x has exactly five conjugates under $\text{Aut}_{\mathcal{F}}(A)$ by (3.21), which is not the case. So

$$(3.23) \quad h^{\text{Aut}_{\mathcal{F}}(A)} = \{x, h, hz\}.$$

Since Q is cyclic and normal in T and h is not $N_Q(A)$ -conjugate to hx , it follows that $[Q, h] = 1$ and hence

$$(3.24) \quad \Omega_1(T) = \Omega_1(R_d) = \langle x \rangle \times P\langle h \rangle.$$

We claim that

$$(3.25) \quad P\langle h \rangle \text{ is normal in } T.$$

If this does not hold, then from (3.24) and the fact that P is normal in T , there exists $t \in T$ with $h^t \in Phx \cup Phxz$. But all involutions in Ph are P -conjugate (3.22), and hence multiplying t by a suitable element of P , we have that h is $N_T(A)$ -conjugate to hx or to hxz , contradicting (3.23).

We now complete the proof via transfer arguments. Note that F is cyclic or quaternion by Proposition 3.6. If F is cyclic, then as it covers $T/P\langle h \rangle$, we can apply Proposition 1.23 to get that x is \mathcal{F} -conjugate to z , an immediate contradiction to (3.21). So F is quaternion and consequently, $|F : Q| = 2$ by an argument analogous to that of Lemma 1.3(c). Let $w \in F - Q$ of order 4, so that $w^2 = x$. In the present situation, $R_d = Q \times P\langle h \rangle$, w is of least order in $T - R_d$ by (3.24), and T/R_d is cyclic of order 2. So Proposition 1.23 yields a morphism $\varphi \in \mathcal{F}$ with $w^\varphi \in R_d$, and hence $(w^\varphi)^2 \in Z$ by the structure of R_d . As $w^2 = x$, $x^\varphi = x$ by (3.21). Thus $\varphi \in \mathcal{C}$ and so φ extends to a morphism $\tilde{\varphi}$ on a subgroup of T containing Q because $Q \trianglelefteq \mathcal{C}$. This forces $F^{\tilde{\varphi}} \leq \langle Q, w^{\tilde{\varphi}} \rangle$ to be abelian, a final contradiction. \square

4. THE 2-RANK 3 CASE

Continuing the notation from Section 3, we prove here the following reduction.

Theorem 4.1. *Let \mathcal{F} be a fusion system on S satisfying Hypothesis 2.5. Then T is of 2-rank 4.*

Throughout this section, assume to the contrary that T is of 2-rank 3. By Hypothesis 2.5, S is also of 2-rank 3. From Proposition 3.18, we may assume that

$$(4.2) \quad x \notin Z(S).$$

Recall F_1 contains Q with index 1 or 2 and F_1/Q induces field automorphisms on K . Then

$$(4.3) \quad C_T(P) = F_1 \times \langle z \rangle.$$

By Lemma 2.10, there exists no involution in T which is an f -element. Set $J = J(S) = J(T)$ for short. We have the inclusions $P \leq J \leq \Omega_1(T) \leq R_d$. This shows that $Z = \Omega_1(Z(\Omega_1(T))) \leq \Omega_1(C_T(P)) = \Omega_1(F_1 \times \langle z \rangle) = \langle x, z \rangle$. So $Z = \langle x, z \rangle$ and Z coincides with $\Omega_1(Z(J))$. Therefore, since $\text{Baum}(S) \leq T$ and (4.2), we have that $T = \text{Baum}(S)$ and

$$(4.4) \quad T \text{ is of index 2 in } S.$$

Fix $a \in S - T$.

Lemma 4.5. $z \notin x^\mathcal{F} \cup (xz)^\mathcal{F}$.

Proof. Since x is fully \mathcal{F} -centralized and not central in S , we need only show that $z \in Z(S)$. Suppose that $z^a \neq z$ and hence $P^a \neq P$. Since P and P^a are normal in T , we have $[P^a, P] \leq P^a \cap P$ is normal in both P and P^a . Furthermore, $[P^a, P] \neq 1$ since otherwise $P^a P = P^a \times P$ is of 2-rank 4. Therefore $[P^a, P]$ contains both $Z(P^a)$ and $Z(P)$. But $[Z(P^a), P] = 1$, forcing $Z(P^a) = Z(P)$ contrary to assumption. \square

It follows in particular that $x^a = xz$.

Lemma 4.6. $Q = \langle x \rangle$ and F is cyclic.

Proof. Suppose that $Q > \langle x \rangle$ and let $u \in Q$ with $u^2 = x$. Then $\langle u^a \rangle$ is a normal subgroup of T . Since $(u^a)^2 = xz$, we have $[\langle u^a \rangle, T] \leq \langle xz \rangle$. But $[\langle u^a \rangle, P] \leq P$ as P is normal, and it follows that $u^a \in C_T(P) = F_1 \times \langle z \rangle$ by (4.3). Therefore, $xz \in \mathcal{U}^1(\langle u^a \rangle) \leq \mathcal{U}^1(F_1 \times \langle z \rangle) \leq Q$, which is absurd. So $Q = \langle x \rangle$ and consequently F is cyclic (since F/Q is cyclic). \square

Lemma 4.7. *Let V be a four subgroup of P and set $E = \langle x \rangle \times V$. Then E^a is not T -conjugate to E .*

Proof. Suppose that E^a is T -conjugate to E . Modifying a if necessary, we may assume that a normalizes E . Now the subgroup $N = \langle c_a, \text{Aut}_{\mathcal{C}}(E) \rangle$ of $\text{Aut}_{\mathcal{F}}(E)$ lies in $L_3(2)$ and does not act transitively on $\mathcal{I}_2(E)$ by Lemma 4.5. As $x^a = xz$, N does not stabilize a point of E . So N must fix a line, which is then V . It follows that $N = \text{Aut}_{\mathcal{F}}(E) \cong S_4$. Now $|\text{Aut}_T(E)| = 2$, and we can obtain a contradiction to (4.4) by showing that $|\text{Aut}_S(E)| = 8$, i.e. that E is fully automized in \mathcal{F} .

Suppose that E is not fully \mathcal{F} -automized. Either $J = \langle x \rangle \times P$ or there exists an involution $h \in R_d - R$ and $J = \langle x \rangle \times P\langle h \rangle$. In either case, there are exactly two S -classes of elementary abelian subgroups of order 8. Moreover, if $E_1 \in E^{\mathcal{F}}$ is fully \mathcal{F} -automized, then $E_1 \notin E^S$, and so $\langle E, E_1 \rangle = J$. By Alperin's fusion theorem, there is a subgroup $D \in \mathcal{F}^{fc}$ and an automorphism $\alpha \in \text{Aut}_{\mathcal{F}}(D)$ of odd order such that $E_1 := E^\alpha$ is fully \mathcal{F} -automized. But then $J = \langle E, E_1 \rangle \leq D$, and consequently α restricts to a nontrivial (odd order) automorphism of J . On the other hand, $\text{Aut}(J)$ is a 2-group by Lemma 1.4, a contradiction. \square

Lemma 4.8. $R_d > R$.

Proof. Suppose on the contrary that $R_d = R$. If $|T : PF| = 2$, then $J = \langle x \rangle \times P$, and T acts transitively on $\mathcal{E}_{2^3}(T)$ contrary to Lemma 4.7. So $T = PF$. If $|F| > 2$, then $Z(T) = \Omega_2(F) \times \langle z \rangle$ and so $\text{U}^1(Z(T)) = \langle x \rangle$ is normal in S , at odds with (4.2). So $T = R = J = \langle x \rangle \times P$.

We now obtain a contradiction by a transfer argument. Note that as all involutions of Px are \mathcal{F} -conjugate to x , we have $P = \langle z^{\mathcal{F}} \cap T \rangle$ is normal in S by Lemma 4.5. Moreover, the quotient S/P is abelian. If $b \in x^{\mathcal{F}}$ is fully \mathcal{F} -centralized, then $C_S(b)$ has 2-rank 3, whence $b \in T \cap x^{\mathcal{F}} \subseteq Px$. Proposition 1.24 now says that x is \mathcal{F} -conjugate into P , contradicting Lemma 4.5 and completing the proof. \square

As a consequence of the previous lemma, T is transitive on $\mathcal{E}_{2^3}(T)$. Fix a four subgroup V of P . By Lemma 4.7 and the preceding remark, $V^a \in R_d - R$. Fix an involution $h \in V^a - P$. Then $P_1 := P\langle h \rangle$ is dihedral of order $2|P|$, and therefore is generated by \mathcal{F} -conjugates of z . We have at this point that $J = \langle x \rangle \times P_1 = R_d$. As every involution in Phx is P -conjugate to hx , and hence S -conjugate into Px , it follows that

$$(4.9) \quad P_1 = \langle z^{\mathcal{F}} \cap P_1 \rangle = \langle z^{\mathcal{F}} \cap T \rangle.$$

So

$$(4.10) \quad P_1 \text{ is normal in } S.$$

Recall now that $C_T(P) = \Omega_2(F) \times \langle z \rangle$ from (4.3) and Lemma 4.6. Moreover, if $\Omega_2(F) > \langle x \rangle$, then $[\Omega_2(F), h] = \langle z \rangle$ by (4.10) and Lemma 2.1(h). Consequently $C_S(P_1) = Z$ and so

$$(4.11) \quad S/P_1Z \text{ is abelian}$$

by Lemma 1.3 because P_1 is nonabelian dihedral.

Notice FP_1/P_1 is a cyclic normal subgroup of S/P_1 of index 2. So S/P_1 is either abelian or modular, or else $|F| = 4$ by (4.11) and S/P_1 is dihedral or quaternion. We rule out each of these cases in turn.

Lemma 4.12. *S/P_1 is not abelian.*

Proof. Suppose S/P_1 is abelian. For any $b \in x^{\mathcal{F}}$ which is fully \mathcal{F} -centralized, $C_S(b)$ is of 2-rank 3, and so $b \in J \cap x^{\mathcal{F}} \subseteq P_1x$ by (4.9) and Lemma 4.5. Now x has an \mathcal{F} -conjugate in P_1 by Proposition 1.24, and this contradicts Lemma 4.5. \square

The next lemma shows that S/P_1 is not quaternion.

Lemma 4.13. *There exists an involution b in $S - T$.*

Proof. Let $b_1 \in S - T$. Modifying b_1 by an element of F , we may assume $b_1^2 \in P_1Z$ because S/P_1 is not cyclic (by the previous lemma). But then $P_1Z\langle b_1 \rangle / \langle x \rangle$ is dihedral or semidihedral because b_1 swaps the P_1 -classes of four-subgroups of P_1 by Lemma 4.7. Modifying b_1 by an element of P_1 then, we may assume $b_1^2 \in Z$. Now $C_Z(b_1) = \langle z \rangle$ so $b_1^2 \in \langle z \rangle$. Set $b = b_1$ if b_1 is an involution, and set $b = xb_1$ otherwise. Then b is an involution. \square

Lemma 4.14. *S/P_1 is not modular.*

Proof. Suppose it is. Then S/P_1 has a unique four subgroup, covered by $\langle x, b \rangle$. Hence, $S_0 := \Omega_1(S) = P_1Z\langle b \rangle$ and S/S_0 is cyclic. Let $w \in F$ with $w^2 = x$. Then w is of least order outside S_0 and centralizes FP , whence $|S : C_S(w)| \leq 4$. Apply Proposition 1.23 to obtain a morphism φ in \mathcal{F} with w^φ in S_0 and fully \mathcal{F} -centralized.

Any element of $S_0 - P_1Z$ interchanges the two classes of four-subgroups of P_1 . Hence if $b_1 \in S_0 - P_1Z$ is of order 4, then $b_1^2 \in Z$ and b_1 induces an involutory automorphism of P_1 interchanging the two classes of four-subgroups of P_1 . So $C_{P_1}(b_1) = \langle z \rangle$ and $|S : C_S(b_1)| \geq |P_1 : C_{P_1}(b_1)| \geq 8$ as $|P_1| \geq 16$.

As w^φ is fully \mathcal{F} -centralized, the preceding paragraph implies $w^\varphi \in P_1Z = \langle x \rangle \times P_1$, and consequently $w^\varphi = x_0v$ for some $x_0 \in \langle x \rangle$ and $v \in P_1$ of order 4. Now $x^\varphi = (w^\varphi)^2 = z$, contrary to Lemma 4.5. \square

Therefore by the previous three lemmas, $|F| = 4$ and S/P_1 is dihedral of order 8. We now obtain the final contradiction, completing the proof of Theorem 4.1.

Lemma 4.15. *S/P_1 is not dihedral.*

Proof. Suppose it is. Again let $w \in F$ with $w^2 = x$. Then $F = \langle w \rangle$. Since $C_T(F) = R$ is of index 2 and $[F, h] = \langle z \rangle$, we have $F \leq Z_2(T)$ in the present situation. Moreover, $T/Z = F/Z \times P_1Z/Z$ with the second factor dihedral of order at least 8, and so $Z_2(T) = F \times V$ where V is cyclic of order 4 in P_1 . Now b inverts P_1w by assumption; hence $ww^b \in Z_2(T) \cap P_1 = V$. As $[w, w^b] = 1$, we have on the one hand that $ww^b \in C_V(b) = \langle z \rangle$, because $b^2 = 1$, and on the other $(ww^b)^2 = w^2(w^b)^2 = w^2(w^2)^b = x(xz) = z$. These two facts are incompatible, and the proof is complete. \square

5. THE 2-RANK 4 CASE: $|Q| = 2$

For a fusion system \mathcal{F} on S satisfying Hypothesis 2.5, T has 2-rank 3 or 4 by Lemma 2.10. By Theorem 4.1, there are no such fusion systems with T of 2-rank 3. We begin now the study of \mathcal{F} when the rank of T is 4. The content of the current section will be devoted to the proof of the following reduction.

Theorem 5.1. *Assume that \mathcal{F} satisfies Hypothesis 2.5 with T of 2-rank 4. Then $|Q| > 2$.*

The notation follows that begun in Section 2, in particular that of Hypothesis 2.5. For instance, \mathcal{K} is the unique component of the involution centralizer $\mathcal{C} = C_{\mathcal{F}}(x)$, and is a fusion system of a finite group K isomorphic with $L_2(q)$ for suitable $q \equiv \pm 1 \pmod{8}$ (as chosen once and for all after Proposition 2.6). The Sylow subgroup of \mathcal{K} is denoted by P , a dihedral group of order 2^k ($k = \nu_2(q^2 - 1) - 1 \geq 3$). Consistent with Sections 3 and 4, we also set $Z(P) = \langle z \rangle$. Denote by C the cyclic maximal subgroup of P . By Hypothesis 2.5, the Thompson subgroup $J(T) = J(S)$ and so this common subgroup is denoted simply by J .

By Lemma 2.10,

$$(5.2) \quad \text{there exists an involutory } f\text{-element } f \in C_T(P).$$

We fix such an involution f .

Before beginning the proof of Theorem 5.1, we collect some facts seen before, and which hold throughout 2-rank 4 case. In particular,

$$(5.3) \quad J \leq R\langle f \rangle = Q\langle f \rangle \times P$$

from Lemma 2.8, and

$$(5.4) \quad C_T(P) = Q\langle f \rangle \times \langle z \rangle$$

from Lemma 2.7 and the structure of $\text{Aut}(K)$ in Lemma 2.1. Finally,

$$(5.5) \quad T < S$$

by Proposition 3.18.

Assume for the remainder of this section that $Q = \langle x \rangle$ is of order 2, as we prove Theorem 5.1 by way of contradiction in a series of lemmas. Thus, $J = \langle x, f \rangle \times P$ by (5.3) and $Z(J) = \langle x, f, z \rangle$ is elementary abelian of order 8.

Lemma 5.6. $z^{\mathcal{F}} \cap Z(J) = \langle z \rangle$.

Proof. The Thompson subgroup $J = \langle x, f \rangle \times P$ is weakly \mathcal{F} -closed. By Lemma 1.6 fusion in $Z(J)$ is controlled in $\text{Aut}_{\mathcal{F}}(J)$. But $\mathcal{U}^1(J) \cap Z(J) = \langle z \rangle$ is characteristic in J , so the statement follows. \square

Lemma 5.7. *Let $y \in Z(J)$. Then each involution of yP is \mathcal{C} -conjugate to y or to yz . In particular, $z^{\mathcal{F}} \cap J = z^{\mathcal{F}} \cap P$.*

Proof. Since \mathcal{K} has one class of involutions and $x \in C_T(\mathcal{K})$ the lemma holds for $y = x$, z , and xz . So we may assume that y is an f -element on \mathcal{K} , that is, y centralizes P but $y \notin \langle x, z \rangle$. Let t be an involution of P so that yt is also an involution. If $t = z$ then the statement is obvious, so assume t is a noncentral involution of P . Set $U = \langle t, z \rangle$,

and let $\varphi \in \text{Aut}_{\mathcal{K}}(U)$ of order 3 such that $t^\varphi = z$. Then φ extends to $\tilde{\varphi} \in \mathcal{C}$ on $U\langle y \rangle$ and centralizes either y or yz by Lemma 2.8. In the former case, $(yt)^{\tilde{\varphi}} = yz$, and in the latter, $(yt)^{\tilde{\varphi}^2} = (yztz)^{\tilde{\varphi}^2} = y$. This completes the proof of the first statement. The second statement now follows from Lemma 5.6. \square

Lemma 5.8. *The following hold.*

- (a) P is normal in S ,
- (b) there exists a fully \mathcal{F} -centralized four subgroup of P ,
- (c) $[S, S] \leq C_S(C)$, and
- (d) no element of S squares into $J - Z(J)C$.

Proof. Let $s \in S$. Then $z^s \in Z(J)$, and so $z^s = z$. But all involutions of P^s are \mathcal{K}^s -conjugate by Lemma 2.1(f). Hence $P^s = \Omega_1(P^s) = \langle z^{\mathcal{K}^s} \rangle \leq P$ by Lemma 5.7 and (a) holds.

Let U be a four subgroup of P , and let $\psi \in \text{Hom}_{\mathcal{F}}(U, S)$ such that U^ψ is fully \mathcal{F} -centralized. By the extension axiom, ψ extends to $C_S(U)$, which contains an elementary abelian subgroup of maximal rank. Thus, $U^\psi \leq J$ and the nonidentity elements of U^ψ consist of \mathcal{F} -conjugates of z . It follows that $U^\psi \leq P$ by Lemma 5.7, proving (b).

Part (c) is Lemma 1.3(b). For (d), suppose that $s \in S$ with $s^2 \in J - Z(J)C$. Then s^2 must lie in $C_S(P)C$ by Lemma 1.3(c). Thus, $s^2 \in C_J(P)C = Z(J)C$. \square

Lemma 5.9. $J = PC_S(P)$.

Proof. Suppose the lemma is false and choose $a \in C_S(P) - J$ with $a^2 \in J$. Let $Z_a = C_{Z(J)}(a)$, which contains z and is of order 4. Then $a^2 \in Z(J) \cap C_S(a) = Z_a$ since a centralizes P . By (5.4) and (5.3), $J = PC_T(P)$, and so a does not centralize x .

Fix a fully \mathcal{F} -centralized four subgroup U of P guaranteed by Lemma 5.8(b), a \mathcal{K} -automorphism φ of U of order 3, and an extension $\tilde{\varphi} \in \text{Hom}_{\mathcal{F}}(C_S(U), S)$ (by the extension axiom). Observe that

$$(5.10) \quad Z(J)a \text{ contains no involution,}$$

because each element of $Z(J)a$ lies outside J and centralizes the subgroup $Z_a U$, which is of 2-rank 3. Also, $\tilde{\varphi}$ is defined on a ; it follows that

$$(5.11) \quad a^2 \neq z,$$

since otherwise $a^{\tilde{\varphi}}$ is an element of S squaring to a noncentral involution of P , contrary to Lemma 5.8(d).

If $[x, a] = z$, then $[x^{\tilde{\varphi}}, a^{\tilde{\varphi}}] = z^{\varphi}$ is a noncentral involution of P , contradicting Lemma 5.8(c). Finally, we consider the case in which $[x, a] = y \neq z$. Here, $a^2 \in Z_a = \langle y, z \rangle$. If $a^2 = y$, then x inverts a , and so $(xa)^2 = 1$ contrary to (5.10). Hence $a^2 = yz$ by (5.11). In this case, we may replace a by xa to obtain $a^2 = z$, again contradicting (5.11), and completing the proof. \square

Let Ω be the two element set consisting of the P -classes of four-subgroups of P . Let N be kernel of the action of S on Ω . Then $J \leq N$. By the previous two lemmas S/J embeds into $\text{Out}(P)$. Thus by Lemma 1.3(a), S/J has a cyclic subgroup B with index 1 or 2 and with $B = N/J$ cyclic of order dividing 2^{k-3} . Thus, N is of index 1 or 2 in S .

Lemma 5.12. *We have $J < N$. In particular, $|P| \geq 16$.*

Proof. Suppose to the contrary that $J = N$. Then $N \leq T$ and $T < S$ by (5.5). Since $|S : N| \leq 2$, it follows that $N = J = T$ and $|S : T| = 2$.

Fix $a \in S - T$. As a acts on $Z(J) = Z(T)$ and does not centralize x , $Z(S) = \langle y, z \rangle$ for some $y \in Z(J) - \langle x, z \rangle$. Since a normalizes P and acts nontrivially on Ω , we have $[P, a] = C$. Thus, we have two possibilities for the commutator subgroup of S . Either $[x, a] = z$ and $[S, S] = C$, or else $[S, S] = Z(S)C$.

Assume first that $[S, S] = C$. We have that $y \in Z(S) - \langle z \rangle$ is not \mathcal{F} -conjugate to z by Lemma 5.6. But then Proposition 1.24, applied with C representing the T of that Proposition, forces $z \in y^{\mathcal{F}}$ anyway, a contradiction.

Now suppose $[S, S] = Z(S)C$, so that $[S, S] \cap Z(J) = Z(S)$ is of order 4. We will show in this case that $x \notin \text{foc}(\mathcal{F})$. We claim

$$(5.13) \quad \text{every fully } \mathcal{F}\text{-centralized conjugate of } x \text{ lies in } Z(J).$$

Let s be a fully \mathcal{F} -centralized conjugate of x . Then s lies in a elementary abelian subgroup of rank 4 by the extension axiom, so $s \in J = Z(J)P$. If $s \in Z(J)(P - C)$, then s has at least four conjugates under $P\langle a \rangle$ because $Z(J) \cap P = \langle z \rangle$ and $P\langle a \rangle$ is transitive on the involutions in $P - C$. So $|C_S(x)| > |C_S(s)|$, and (5.13) holds.

Thus, (5.13) implies $Z(S)Cx$ is the unique nonidentity element of $S/Z(S)C$ containing a fully \mathcal{F} -centralized \mathcal{F} -conjugate of x . This allows us to apply Proposition 1.24, with $Z(S)C = [S, S]$ in the role of T , to obtain an \mathcal{F} -conjugate of x in $\Omega_1([S, S]) = Z(S)$, contradicting the assumption that x is fully \mathcal{F} -centralized. We conclude that $J < N$, and the first statement of the lemma holds.

For the last statement, suppose $|P| = 8$. Every element inducing an outer automorphism on P interchanges the two classes of four-subgroups of P . Thus N induces inner automorphisms on P , i.e. $N = PC_S(P) = J$, contrary to $J < N$. Therefore, $|P| \geq 16$. \square

By an earlier remark and Lemma 5.12, N/J is nontrivial cyclic. Choose $w \in N$ mapping to a generator of N/J . By the definition of N , we may adjust w by an element of P and assume that w centralizes a four subgroup $U = \langle e, z \rangle$ of P . Replacing w by ew if necessary, we may assume also that

$$(5.14) \quad w \text{ centralizes } C/\mathcal{U}^2(C).$$

Let $f_1 \in \langle w \rangle$ such that $f_1 \notin J$ but $f_1^2 \in J$. Since $J = PC_S(P)$ by Lemma 5.9, we have $f_1^2 \in C_S(C)$ by Lemma 1.3(d) applied with $D = P$ there. Then f_1 takes a generator c of C to cz by choice of $f_1 \in \langle w \rangle$ and (5.14), and hence f_1 centralizes $\mathcal{U}^1(C)$. As $|P| \geq 16$ from Lemma 5.12, it follows that

$$(5.15) \quad C_P(f_1) \text{ is the nonabelian dihedral subgroup } P_1 := \langle U, \mathcal{U}^1(C) \rangle \text{ of } P.$$

So $f_1^2 \in C_S(P_1) \cap J = Z(J)$, and f_1 is of order at most 4. But f_1 is not an involution, otherwise $\langle f_1, C_{Z(J)}(f_1), U \rangle$ is an elementary 16 outside J , so f_1 is of order 4. In fact, it is shown below that there are no involutions in Jf_1 . For this, we will need that

$$(5.16) \quad f_1 \text{ does not square to } z.$$

Assume to the contrary that $f_1^2 = z$. Then as $[\mathcal{U}^1(C), f_1] = 1$ and $|P| \geq 16$ by Lemma 5.12, there exists an element $v \in C_P(f_1)$ with $(vf_1)^2 = 1$. But then $C_P(vf_1)$ contains the four subgroup $\langle ce, z \rangle$ of P , and so $vf_1 \in J$, yielding the same contradiction as before and thus confirming (5.16). We now show

Lemma 5.17. *There is no involution in Jf_1 .*

Proof. By (5.16), we may assume that $f_1^2 \in Z(J) - \langle z \rangle$. If $[Z(J), f_1] \leq \langle z \rangle$, then $J\langle f_1 \rangle/P \cong C_2 \times C_4$ with Pf_1 of order 4. In this case, every element of order 2 in $J\langle f_1 \rangle$ lies in J as claimed. Hence we may assume that $[Z(J), f_1] = \langle y \rangle \neq \langle z \rangle$. Then $Z(J)P_1\langle f_1 \rangle = \langle y_1, f_1 \rangle \times P_1$ with $D_1 := \langle y_1, f_1 \rangle \cong D_8$. If E_1 is the other four subgroup of D_1 , then $E_1 \times P_1$ has 2-rank 4 and so $E_1 \leq J$. But then $D_1 \leq C_S(P_1) \cap J = Z(J)$, a contradiction. \square

In addition, we let $h_1 \in S - N$ be an element such that $h_1^2 \in J$ or set $h_1 = 1$ if such an element does not exist. Note that if $h_1 = 1$, then S/J is cyclic by the structure of $\text{Out}(P)$ (Lemma 1.3(a)). In any case, $S/J\langle h_1 \rangle$ is cyclic.

Assume that $h_1 \neq 1$. Then

$$(5.18) \quad \text{both } h_1 \text{ and } h_1f_1 \text{ square into } J.$$

Let $s \in Jh_1 \cup Jh_1f_1$. Then for any $e_1 \in P - C$, we have $e_1^s = e_1c$ for some generator c of C because $s \notin N$. Therefore,

$$(5.19) \quad [P, s] = C \text{ and } C_J(s) \leq Z(J)C \text{ is abelian.}$$

Furthermore, as $e_1^{s^2} = e_1cc^s$, we have

$$(5.20) \quad \text{if } s^2 \in C_S(P) \text{ then } s \text{ inverts } C.$$

With this setup, the next two lemmas contradict each other and complete the proof of Theorem 5.1.

Lemma 5.21. *$h_1 \neq 1$ and both cosets Jh_1 and Jh_1f_1 contain involutions.*

Proof. Suppose either that $h_1 = 1$ or that there are no involutions in Jh_1f_1 . The argument is the same in case $h_1 \neq 1$ and Jh_1 contains no involutions. (Alternatively, swap the roles of h_1 and h_1f_1 in this extra case.) By (5.17) and assumption, $\Omega_1(S) \leq J\langle h_1 \rangle$. Also $S/J\langle h_1 \rangle$ is cyclic, and f_1 is of least order outside $J\langle h_1 \rangle$. By Proposition 1.23, there exists a morphism $\varphi \in \mathcal{F}$ such that $f_1^\varphi \in J\langle h_1 \rangle$ is fully \mathcal{F} -centralized, and $C_S(f_1)^\varphi \leq C_S(f_1^\varphi)$.

Now if $h_1 \neq 1$, then f_1^φ cannot lie in the coset Jh_1 . This is because $\Omega_1(C_P(f_1))$ is nonabelian dihedral by (5.15), whereas $\Omega_1(C_S(s))$ is abelian for every $s \in Jh_1$ by (5.19). So $f_1^\varphi \in J$ whether or not $h_1 = 1$. Since f_1 is of order 4 and $\Omega_1(\mathcal{U}^1(J)) = \langle z \rangle$, we have that $(f_1^\varphi)^2 = z$. But z is weakly \mathcal{F} -closed in $Z(J)$ by Lemma 5.6 and so $f_1^2 = z$, contrary to (5.16). \square

Lemma 5.22. *Jh_1f_1 contains no involution.*

Proof. We may assume $h_1 \neq 1 = h_1^2$ by Lemma 5.21, and then h_1 inverts C by (5.20). So

$$(5.23) \quad h_1f_1 \text{ sends a generator } c \text{ of } C \text{ to } c^{-1}z.$$

In particular, $C_C(h_1f_1) = \langle z \rangle$ and so $C_{Z(J)C}(h_1f_1) \leq Z(J)\Omega_2(C)$. As $(h_1f_1)^2 \in J$ from (5.18), we have $(h_1f_1)^2 \in C_J(h_1f_1) = C_{Z(J)C}(h_1f_1) \leq Z(J)\Omega_2(C)$ with the equality by (5.19). But $(h_1f_1)^2$ does not lie in $C_S(P) = Z(J)$ by (5.20), since h_1f_1 does not invert C .

Set $M = J\langle h_1f_1 \rangle$ and $\overline{M} = M/Z(J)$. Then \overline{M} contains the dihedral group \overline{P} as a maximal subgroup, which is nonabelian as $|P| \geq 16$. Furthermore \overline{M} is of maximal class by (5.19) and (5.23). As $\overline{h_1f_1}$ is of order 4 squaring into the center of \overline{M} , we know \overline{M} is semidihedral. But then \overline{M} contains no involutions outside its dihedral maximal subgroup \overline{P} . It follows that $M = J\langle h_1f_1 \rangle$ contains no involutions outside J , which is what was to be shown. \square

6. THE 2-RANK 4 CASE: $|Q| > 2$

For this final section, we continue to assume \mathcal{F} is a saturated fusion system on the 2-group S satisfying Hypothesis 2.5. By the main results of the previous three sections, we are reduced to the following situation in describing \mathcal{F} .

- (1) $T = C_S(x)$ is a proper subgroup of S (Proposition 3.18),
- (2) S is of 2-rank 4 (Theorem 4.1), and
- (3) $Q = C_T(\mathcal{K})$ is of order at least 4 (Theorem 5.1).

Theorem 6.1. *Let \mathcal{F} be a saturated fusion system on the 2-group S . Assume \mathcal{F} satisfies Hypothesis 2.5 and, in addition, the above three items. Then $S \cong D_{2^k} \wr C_2$, and \mathcal{F} is the fusion system of $L_4(q)$ for some $q \equiv 3 \pmod{4}$ with $\nu_2(q+1) = k-1$.*

Adopt the notation of Hypothesis 2.5 and the set up at the beginning of Section 5. By Lemma 2.10,

there exists an involutory f -element $f \in C_T(P)$.

We continue to fix such an involution f . As T a proper subgroup of S , we also

$$\text{fix } a \in N_S(T) - T \text{ with } a^2 \in T.$$

As usual, we prove Theorem 6.1 in a sequence of lemmas. It will emerge quickly (after Lemma 6.8) that $J = R\langle f \rangle$ is the product of two dihedral groups $Q\langle f \rangle$ and P of the same order, T has index 2 in S , and T/R is of exponent 2. Since T/R embeds into $\text{Out}(K)$ (Proposition 2.7), this means that either $T = J = R\langle f \rangle$, or $T = R\langle h, f \rangle$ for some $1 \neq h \in T$ such that $h^2 \in Q$ and $R\langle h \rangle/Q$ is dihedral of order $2|P|$. Thus $R\langle h \rangle\mathcal{K}/Q$ is uniquely determined as the fusion system of $PGL_2(q)$ (see the description of $\text{Out}(K)$ in Lemma 2.1). In anticipation of this we let

$$(6.2) \quad \begin{aligned} &h \in T - R \text{ such that } h^2 \in Q \text{ and } R\langle h \rangle/Q \text{ is dihedral, or} \\ &h = 1 \text{ if such an element does not exist.} \end{aligned}$$

Note in the case $h \neq 1$,

$$(6.3) \quad [Qh, Qf] = Qz$$

by Lemma 2.1(h).

Much of the 2-group and transfer analysis will be dedicated to analyzing whether or not h exists, and if it does, whether $Q\langle h \rangle$ splits over Q . Together with the target $\mathcal{F} = \mathcal{F}_S(L_4(q))$ appearing within the case $T = J$, the following table lists the fusion systems (of finite

groups) which nearly satisfy the conditions of Theorem 6.1, and why they are eventually ruled out.

| Scenario | T | S | Group | Contradiction |
|---|----------------------|------------------------|------------|---|
| $h \neq 1$ and $Q\langle h \rangle$ is dihedral | $J\langle h \rangle$ | $Q_{2^{k+1}} \wr C_2$ | $PSp_4(q)$ | $C_T(\mathcal{K}) \cong D_{2^k}$ |
| $h \neq 1$ and $Q\langle h \rangle$ is cyclic | $J\langle h \rangle$ | $SD_{2^{k+1}} \wr C_2$ | $PGL_4(q)$ | $O^2(\mathcal{F}) < \mathcal{F}$ |
| $h = 1$ | J | $D_8 \wr C_2$ | A_{10} | $C_T(\mathcal{K}) \cong C_2 \times C_2$ |
| $h = 1$ | J | $D_{2^k} \wr C_2$ | $L_4(q)$ | |

The 2-group and transfer analysis is carried out through Lemma 6.23, where it is shown that S is of type $PSp_4(q)$ or $PGL_4(q)$ when $h \neq 1$. Then we compute the centralizer of a central involution via an argument modeled on that of [GH73, Lemmas 3.15, 3.16], thus ruling out the $PGL_4(q)$ -case. Analyzing the resulting fusion information allows us to conclude that $h = 1$, and S is then isomorphic to $D_{2^k} \wr C_2$. Lastly, we appeal to a result of Oliver [Oli] to identify \mathcal{F} as the fusion system of $L_4(q)$.

We begin by pinning down the structure of J in the next few lemmas.

Lemma 6.4. *The following hold.*

- (a) $Q\langle f \rangle$ is dihedral or semidihedral,
- (b) $J = \Omega_1(Q\langle f \rangle) \times P$,
- (c) no element of Rf is a square in T , and
- (d) $T = R\langle h, f \rangle$.

Proof. We claim that $Z(Q\langle f \rangle) = \langle x \rangle$. Suppose this is not the case. Then either f centralizes Q or $|Q| \geq 8$ and f acts on Q by sending a generator d of Q to dx . In either case we have that $J = \langle x, f \rangle \times P$ by (5.3). Then $D = C_T(J)$ is normal in S , and D is equal to $Q \times \langle f, z \rangle$ if f centralizes Q and to $\Omega^1(Q) \times \langle f, z \rangle$ otherwise. In any case, x is the only involution which is a square in D , so $x \in Z(S)$. This contradicts $T < S$. Thus $Q\langle f \rangle$ is of maximal class and f is an involution outside Q , so (a) holds. Now (b) follows by (5.3).

Suppose some element $f_1 \in T$ squares into Rf . Write $\tilde{T} = T/P$ and denote images modulo P similarly. Then $D_1 := \tilde{Q}\langle \tilde{f}_1 \rangle$ contains $D := \tilde{Q}\langle \tilde{f} \rangle$ as a normal subgroup. Moreover as $Q\langle f \rangle \cap P = 1$, $D \cong Q\langle f \rangle$. If $Q\langle f \rangle$ is dihedral in (a), then Lemma 1.3(d), with D_1 in the role of S there, gives a contradiction. If $Q\langle f \rangle$ is semidihedral, then considering the images of D_1 and D modulo $\langle \tilde{x} \rangle$, we obtain the same contradiction to Lemma 1.3(d). Therefore, (c) holds. By the structure of $\text{Out}(K)$ and (c), T/R is a four group covered by $\langle h, f \rangle$, yielding (d). \square

Lemma 6.5. $x^S = \{x, z\}$. Consequently, $|S : T| = 2$.

Proof. By Lemma 6.4, S normalizes $Z(J) = \langle x, z \rangle$ and $J = J(Q\langle f \rangle) \times P = \Omega_1(Q\langle f \rangle) \times P$ is the product of two nonabelian dihedral groups. By Proposition 1.2, S must permute the commutator subgroups of these factors. Therefore, $x^S \subseteq \{x, z\}$ and as $T = C_S(x)$ is a proper subgroup of S , we have equality and $|S : T| = 2$. \square

Recall $a \in S - T$ has been fixed, squaring into T . By the previous lemma, $T \trianglelefteq S = T\langle a \rangle$ and a swaps x and z . So $Z(S) = \langle xz \rangle$, and

$$(6.6) \quad x \text{ is not } \mathcal{F}\text{-conjugate to } xz$$

because $x \notin Z(S)$ is fully \mathcal{F} -centralized.

Lemma 6.7. $[P^a, P] = P^a \cap P = 1$.

Proof. Note first that both P and P^a are normal in T , So $[P, P^a] \leq P \cap P^a$. Suppose that $Z_0 := P \cap P^a \neq 1$. Then Z_0 is nontrivial normal in P^a , and so $\langle x \rangle = Z(P^a) \leq Z_0 \leq P$, a contradiction. \square

Lemma 6.8. $Q\langle f \rangle$ is dihedral of the same order as P .

Proof. Recall from Lemma 6.4 that $\Omega_1(Q\langle f \rangle)$ is nonabelian dihedral and $J = \Omega_1(Q\langle f \rangle) \times P$. Let C_1 and C_2 be the cyclic maximal subgroups of $\Omega_1(Q\langle f \rangle)$ and P , respectively. Because $x^a = z$, we have C_1^a is a cyclic subgroup of J with z as its unique involution, and so $|C_1| \leq |C_2|$ by the structure of J . We conclude similarly that $|C_2| \leq |C_1|$ by considering C_2^a , and hence C_1 and C_2 are of the same order. Therefore either $Q\langle f \rangle$ is either dihedral of the same order as P , or $Q\langle f \rangle$ is semidihedral with $|Q\langle f \rangle| = 2|P|$.

Suppose $Q\langle f \rangle$ is semidihedral. Then $\Omega_1(Q\langle f \rangle) \cong P$. Set $S_1 = (Q\langle f \rangle)^a$ for short. Then S_1 centralizes P^a . But P also centralizes P^a by Lemma 6.7. In fact,

$$P^a \times \langle z \rangle \leq \Omega_1(C_T(P)) = \Omega_1(Q\langle f \rangle) \times \langle z \rangle$$

so the above three subgroups are equal, as the outside two are of the same order. Taking centralizers, we get that

$$(6.9) \quad S_1 \leq C_T(P^a) = C_T(\Omega_1(Q\langle f \rangle)) \leq C_T(f)$$

Now f centralizes no element in the coset Rh when $h \neq 1$ by (6.3). So $C_T(f) = C_{R\langle f \rangle}(f) = P \times \langle f, x \rangle$ by Lemma 6.4(d,a), and (6.9) is a contradiction because $P \times \langle f, x \rangle$ contains no semidihedral subgroup. \square

In view of the previous lemma, it is now determined that

$$(6.10) \quad J = Q\langle f \rangle \times P = P^a \times P$$

with a interchanging P and P^a . Since $P^a \leq C_T(P) = Q\langle f \rangle \times \langle z \rangle$, we may replace f by fz and assume that

$$(6.11) \quad f \in P^a.$$

We fix notation for the maximal cyclic subgroup of P , calling it C . Then

$$(6.12) \quad C^a \leq Q\langle z \rangle \text{ and } Q^a \leq C\langle x \rangle$$

by the above remarks.

The next lemma shows that a may be chosen to be an involution. Part (a) of it will later be shown in Lemma 6.25 to rule out the $PSp_4(q)$ -case mentioned above and determine that in fact $h = 1$.

Lemma 6.13. *The following hold.*

- (a) If $C = \langle c \rangle$, then ff^a is not \mathcal{F} -conjugate to $f(cf^a)$.
- (b) There exists an involution in $S - T$.

Proof. We will show that ff^a is not conjugate to $f(cf^a)$ from the fact that one of them is \mathcal{F} -conjugate to x and the other to xz . Recall from (6.11) we have chosen $f \in P^a$, so $f^a \in P$. Thus, $U_0 = \langle f^a, z \rangle$ and $U_1 = \langle cf^a, z \rangle$ are four-subgroups of P which are not P -conjugate. Since f is an f -element on \mathcal{K} (Definition 2.9), $C_C(f)$ contains $\text{Aut}_{\mathcal{K}}(U_j)$ for some j , and $C_C(fz)$ contains $\text{Aut}_{\mathcal{K}}(U_{1-j})$ by Lemma 2.8(b). Thus, there is an element $\varphi \in \text{Aut}_{\mathcal{C}}(\langle f \rangle U_j)$ of order 3 with $(f \cdot c^j f^a)^\varphi = f^\varphi (c^j f^a)^\varphi = fz$. On the other hand, there is a similar element $\psi \in \text{Aut}_{\mathcal{C}}(\langle f \rangle U_{1-j})$ with $(fz \cdot zc^{1-j} f^a)^\psi = fz \cdot z = f$. Since f is \mathcal{K}^a -conjugate to x in $C_{\mathcal{F}}(z)$, and so fz is $C_{\mathcal{F}}(z)$ -conjugate to xz , this contradicts (6.6).

For (b), suppose that $a^2 \in J$. It will be shown first that (b) holds in this situation. Write $a^2 = ts^{-1}$ with $t \in P^a$ and $s \in P$. Let $a_0 = as$. Then $P^{a_0} = P^a$, and $a_0^2 = a^2 s^a s = ts^a$ as $[P, P^a] = 1$. So $a_0^2 \in P^a$ and centralizes a_0 . Therefore $a_0^2 = 1$, as claimed.

So it remains to prove that $a^2 \in J$. If a does not square into J , then $h \neq 1$ and a squares into the coset Jh ; so S/J is cyclic of order 4 in the present case. Let \mathcal{J} denote the set of J -classes of “noncentral diagonal” involutions of J , that is, those involutions in J outside the set $I = P\langle x \rangle \cup P^a\langle z \rangle$. Thus \mathcal{J} has cardinality 4, and for any generator c of C , the set $\{ff^a, (c^a f)f^a, f(cf^a), c^a f(cf^a)\}$ is a set of representatives for the members of \mathcal{J} . Since I is a normal subset of S and J is a normal subgroup, S acts on \mathcal{J} by conjugation. Moreover, any element in Jh swaps the two P -classes of noncentral involutions in P , and so acts nontrivially on \mathcal{J} . It follows that $\langle a \rangle$ acts transitively as a four-cycle on \mathcal{J} , and hence all involutions in $J - I$ are S -conjugate. This contradicts part (a) and completes the proof of the lemma. \square

From now on, we assume $a^2 = 1$. We narrow down the structure of T to two possibilities in the next lemma, depending on whether $Q\langle h \rangle$ splits over Q or not, as described in the introduction to this section.

Lemma 6.14. *Suppose $h \neq 1$. Then one of the following holds.*

- (a) $h^2 = 1$ and $Q\langle h \rangle$ is dihedral, or
- (b) $Q = \langle h^2 \rangle$.

Proof. Recall that $Q \trianglelefteq T$ and $h^2 \in Q$ by the choice of h . Since $T = R\langle h, f \rangle$ and $J = R\langle f \rangle$, it follows that $h^a \in Jh$. The coset $P^a h^a$ lies outside the dihedral group J/P^a , which is isomorphic to P . And T/P^a is a dihedral group containing J/P^a as a maximal subgroup. Thus either $P^a h^a$ is an involution in $T/P^a - J/P^a$, or $P^a h^a$ squares to a generator of the cyclic maximal subgroup CP^a/P^a of J/P^a .

Suppose that $P^a h^a$ is an involution in $T/P^a - J/P^a$. Then $P^a h^a$ inverts $CP^a/P^a \cong C$, and so h^a inverts C . It follows that h inverts $C^{a^{-1}} \leq Q \times \langle z \rangle$. Since h normalizes Q , h must invert Q . As $P^a h^a$ is an involution, we have $(h^a)^2 \in P^a$. So $h^2 \in P$. But $h^2 \in Q$ by choice of h . Therefore, $h^2 \in Q \cap P = 1$ giving (a).

Suppose that $P^a h^a$ squares to a generator of CP^a/P^a . Then h^a and hence h has order at least $2|C|$. But $h^2 \in Q$ and $|Q| = |C|$, so we must have $Q = \langle h^2 \rangle$, giving (b). \square

Set $J_0 = QC = Q \times C = C^a \times C$, a homocyclic normal subgroup of S . It will be helpful for what follows to call attention to the action of T on J_0 , and describe what this means for

the structure of the quotient S/J_0 . Recall that $C^a \leq Q\langle z \rangle$ from (6.12), and so the action of an element of T on C^a is the same as on Q . From Lemma 6.8 and the two possibilities in Lemma 6.14, each element in T centralizes C or inverts it, and the same holds for C^a in place of C . Conjugation by an element in $S - T$ swaps the actions. Moreover, T/J_0 is elementary abelian of order 8 when $h \neq 1$, and a induces an automorphism of T/J_0 fixing pointwise a four group. For instance, from the actions of h , f , and f^a on J_0 , and since $J \leq S$,

$$(6.15) \quad \text{if } h \text{ inverts } Q, \text{ then } \langle h, f f^a \rangle \text{ covers } C_{T/J_0}(a).$$

and

$$(6.16) \quad \text{if } h \text{ centralizes } Q, \text{ then } \langle f h, f f^a \rangle \text{ covers } C_{T/J_0}(a),$$

Lastly,

$$(6.17) \quad [S, S] = J_0 \langle f f^a \rangle.$$

Suppose h is involution as in Lemma 6.14(a) from now through the next lemma. From (6.15), we have $[h, a] \in J_0 = C^a C$ and we may arrange to have $[h, a] \in C^a$ by replacing h by an appropriate element in $C^a h \leq Q\langle z \rangle h$. Then

$$[h, a] = (ha)^2 \in C^a \cap C_{J_0}(ha) = 1,$$

and h still squares to the identity. Fix this choice for h now through the next lemma. Thus, S is a split extension of J by the four group $\langle h, a \rangle$.

We can now write down a presentation for S . The rest of the following lemma is verified by direct computation, or by appeal to [GH73, Lemma 3.5].

Lemma 6.18. *Suppose h is an involution. Then S has presentation*

$$\begin{aligned} \langle d, c, f, e, h, a \mid & d^{2^{k-1}} = c^{2^{k-1}} = f^2 = e^2 = a^2 = 1, \\ & [d, c] = [f, e] = 1, \\ & d^f = d^{-1}, c^e = c^{-1}, c^a = d, e^a = f, \\ & h^2 = 1, e^h = ec, h^a = h \rangle \end{aligned}$$

with notation consistent with that fixed. Here, $P = \langle c, e \rangle$, $P^a = \langle d, f \rangle$, $J = P^a P$, $J_0 = \langle d, c \rangle$, $x = d^{2^{k-2}}$, $z = c^{2^{k-2}}$, $Z(S) = \langle xz \rangle$, and $T = \langle d, c, f, e, h \rangle$. Furthermore, the following hold.

- (a) $J\langle a \rangle \cong D_{2^k} \wr C_2$.
- (b) $Q = \langle dz \rangle$.
- (c) h inverts J_0 and all involutions of Jh are J -conjugate.
- (d) $C_S(h) = \langle h \rangle \times B_0$ where $B_0 = \langle x, a \rangle$ is dihedral of order 8.
- (e) $C_S(a) = \langle a \rangle \times B_a$ where $B_a = \langle f f^a h, h \rangle$ is dihedral of order 2^{k+1} .
- (f) $C_S(ha) = \langle ha \rangle \times B_{ha}$ where $B_{ha} = \langle [h, f] f f^a h, h \rangle$ is dihedral of order 2^{k+1} .
- (g) All involutions of Ja are J -conjugate as are all involutions of Jha .
- (h) $C_S(f f^a) = \langle f, x, a \rangle \cong (C_2 \times C_2) \wr C_2$ and all elements of $J_0 f f^a$ are S -conjugate.

- (i) $D_1 := \langle f f^a h, x f f^a a \rangle$ and $D_2 := \langle [h, f] f f^a h, h \rangle$ are quaternion of order 2^{k+1} with $[D_1, D_2] = 1$, $D_1 \cap D_2 = \langle xz \rangle$, $D_1^f = D_2$, $D := D_1 D_2 = J_0 \langle f f^a, h, a \rangle = [S, S] \langle h, a \rangle$, and $S = D \langle f \rangle \cong Q_{2^{k+1}} \wr C_2$ is of type $PSp_4(q)$.

From now through the next lemma, assume $Q = \langle h^2 \rangle$ as in Lemma 6.14(b). We adjust h slightly as follows. As a consequence of (6.17), we have that $h^a \in J_0 f f^a h = C C^a f f^a h$. Since $a^2 = 1$, we may write $h^a = c_1^a c_1 f f^a h$ for some $c_1 \in C$. Replacing h by $(c_1^a)^{-1} h$, which lies in $Q \langle z \rangle h$ by (6.12), we arrange that

$$(6.19) \quad h^a = f f^a h,$$

and h still squares to a generator of Q . Fix this choice of h once and for all. Then it follows that

$$(6.20) \quad [fh, a] = 1.$$

Now fix the generator $c = [f^a, h]$ of C and set $d = c^a$. Then

$$(6.21) \quad d = c^a = [f, h^a] = [f, f f^a h] = [f, h].$$

From (6.20) and the fact that $(fh)^2 \in Qz$ in (6.3), we have $(fh)^2 = xz$, and so $xz = fhfh = h^{-2}[f, h] = h^{-2}d$ as f inverts Q . Hence,

$$(6.22) \quad h^2 = dxz.$$

Lastly, from $(fh)^2 = xz = (xa)^2$ and (6.20), we have

$$xfha \in Jha \text{ is an involution.}$$

We can now write down a presentation for S in case $Q = \langle h^2 \rangle$. The rest of the following lemma is similarly verified by direct computation.

Lemma 6.23. *Suppose $Q = \langle h^2 \rangle$. Then S has presentation*

$$\begin{aligned} \langle d, c, f, e, h, a \mid & d^{2^{k-1}} = c^{2^{k-1}} = f^2 = e^2 = a^2 = 1, \\ & [d, c] = [f, e] = 1, \\ & d^f = d^{-1}, \quad c^e = c^{-1}, \quad c^a = d, \quad e^a = f \\ & h^2 = dd^{2^{k-2}}c^{2^{k-2}}, \quad e^h = ec, \quad h^a = feh \rangle \end{aligned}$$

with notation consistent with that fixed. Here $P = \langle c, e \rangle$, $P^a = \langle d, f \rangle$, $J = PP^a$, $J_0 = \langle d, c \rangle$, $x = d^{2^{k-2}}$, $z = c^{2^{k-2}}$, $Z(S) = \langle xz \rangle$, and $T = \langle d, c, f, e, h \rangle$. Furthermore, the following hold.

- (a) $J \langle a \rangle \cong D_{2^k} \wr C_2$.
- (b) $Q = \langle dxz \rangle = \langle dz \rangle$.
- (c) There are no involutions in Jh .
- (d) fh inverts J_0 ; all elements of $J_0 fh$ square to xz and are J -conjugate.
- (e) $C_S(a) = \langle a \rangle \times B_a$ where $B_a = \langle f^a h, f f^a \rangle$ is semidihedral of order 2^{k+1} with $Z(B_a) = \langle xz \rangle$.
- (f) Set $b_1 = xfha \in Jha$. Then $b_1^2 = 1$ and $C_S(b_1) = \langle b_1 \rangle \times B_{ha}$ where $B_{ha} = \langle d^{-1} f^a h, xa \rangle$ is semidihedral of order 2^{k+1} with $Z(B_{ha}) = \langle xz \rangle$.
- (g) All involutions of Ja are J -conjugate as are all involutions of Jha .
- (h) $C_S(f f^a) = \langle x, f, a \rangle \cong (C_2 \times C_2) \wr C_2$ and all elements of $J_0 f f^a$ are S -conjugate.

- (i) $D_1 = \langle f^a h, f f^a a \rangle$ and $D_2 = \langle d^{-1} f^a h, a \rangle$ are semidihedral of order 2^{k+1} with $[D_1, D_2] = 1$, $D_1 \cap D_2 = \langle xz \rangle$, $D_1^f = D_2$, $D := D_1 D_2 = J_0 \langle f f^a, f h, a \rangle = [S, S] \langle f h, a \rangle$ and $S = D \langle f \rangle \cong S D_{2^{k+1}} \wr C_2$ is of type $PGL_4(q)$ for $q \equiv 3 \pmod{4}$ with $\nu_2(q+1) = k-1$.

Armed with this data, the centralizer of the central involution is computed next.

Lemma 6.24. *Suppose $h \neq 1$. Then $C_{\mathcal{F}}(xz)$ is realizable by a finite group G having Sylow 2-subgroup S and with the property that G contains a normal subgroup isomorphic to $SL_2(q) * SL_2(q)$ of index 2 with f interchanging the two $SL_2(q)$ factors. In particular, $S \cong Q_{2^{k+1}} \wr C_2$ and h is an involution.*

Proof. Assume that $h \neq 1$. The two possibilities for S in Lemmas 6.18 and 6.23 will be treated simultaneously. Fix $t \in J$ such that tha is an involution as follows. When in the case of Lemma 6.18, we take $t = 1$. In the other case, we take $t = xf$ as in Lemma 6.23(f). In either case th commutes with a and inverts J_0 . Let b be one of a or tha . Then $C_S(b) = \langle b \rangle \times B$ where B is dihedral or semidihedral of order 2^{k+1} with $Z(C_S(b)) \cap [C_S(b), C_S(b)] = Z(S)$ by Lemma 6.18(e,f) and Lemma 6.23(e,f). Moreover, all involutions of Jb are J -conjugate by Lemma 6.18(g) and Lemma 6.23(g).

When h is an involution, all involutions in the coset Jh are J -conjugate by Lemma 6.18(c). From Lemma 6.18(d), $C_S(h) = \langle h \rangle \times B_0$ where B_0 is dihedral of order $8 < 2^{k+1}$, whence $|C_S(h)| < |C_S(b)|$. In the case where $Q = \langle h^2 \rangle$, there are no involutions in Jh by Lemma 6.23(c). This shows that in either case the set of fully \mathcal{F} -centralized \mathcal{F} -conjugates of b outside J lies in $Ja \cup Jha$. By Proposition 1.24, there exists a morphism $\varphi \in \mathcal{F}$ such that $b^\varphi \in J$ is fully \mathcal{F} -centralized, and $C_S(b)^\varphi \leq C_S(b^\varphi)$. Since $C_S(b)$ has nilpotence class $k \geq 3$ and $[S, S, S] \leq J_0$ from (6.17), we have $(xz)^\varphi \in \Omega_1(J_0) = \langle x, z \rangle$. It follows that $(xz)^\varphi = xz$ as xz is not \mathcal{F} -conjugate to x or to z . Composing with c_a if necessary, we may assume that $b^\varphi = z$. Thus, we have shown there exist $\varphi_a, \varphi_{ha} \in C_{\mathcal{F}}(xz)$ such that $a^{\varphi_a} = (tha)^{\varphi_{ha}} = z$.

Set $\mathcal{N} = C_{\mathcal{F}}(xz)$. We claim that the hyperfocal subgroup $\text{hyp}(\mathcal{N})$ is of index 2 in S . We will show this by first demonstrating that the normal closure $\langle [a, \varphi_a]^S, [tha, \varphi_{ha}]^S \rangle$ is the commuting product $D := D_1 * D_2$ of two quaternion or two semidihedral subgroups of order 2^{k+1} as in Lemma 6.18(i) or Lemma 6.23(i), respectively. Then we shall use a transfer argument inside \mathcal{N} to show that in fact $\text{foc}(\mathcal{N}) = D$ from which it will follow that $\text{hyp}(\mathcal{N}) = D$ as well.

Set $D_0 = \langle [a, \varphi_a]^S, [tha, \varphi_{ha}]^S \rangle = \langle (xa)^S, (xtha)^S \rangle$. Taking products, D_0 contains th inverting J_0 . All elements of $J_0 th$ are J -conjugate by Lemma 6.18(c) and Lemma 6.23(d), so D_0 contains J_0 . This shows that $[S, S] = J_0 \langle f f^a \rangle = J_0 \langle [f, a] \rangle \leq D_0$. But D_0 is a proper normal subgroup of S contained in $D = [S, S] \langle th, a \rangle$, and so $D_0 = D$. We conclude that $\text{foc}(\mathcal{N}) \geq D$ is of index 1 or 2 in S .

As f interchanges D_1 and D_2 , $fz \notin D$. Suppose that $\text{foc}(\mathcal{N}) = S$. Then by Proposition 1.23, there exists a morphism $\eta \in \mathcal{N}$ such that $(fz)^\eta \in D$ is fully \mathcal{N} -centralized and $C_S(fz)^\eta \leq C_S((fz)^\eta)$. Since $C_S(fz) = \langle x, f \rangle \times P$ is of 2-rank 4 we have that $(fz)^\eta \in J \cap D = J_0 \langle f f^a \rangle$. Suppose $(fz)^\eta \in J_0 f f^a$. By Lemma 6.18(h) and Lemma 6.23(h) then, $C_S((fz)^\eta) \cong (C_2 \times C_2) \wr C_2$ is of order 2^5 , forcing $|P| = 8$ and $\eta|_{C_S(fz)}$ to be an isomorphism $C_S(fz) \rightarrow C_S((fz)^\eta)$. But $|Z(C_S(fz))| = 8$ whereas $|Z(C_S((fz)^\eta))| = 4$, a contradiction. Therefore $(fz)^\eta \in \Omega_1(J_0) = \langle x, z \rangle$ and $(fz)^\eta = x$ or z because $\eta \in \mathcal{N}$. But

fz is $C_{\mathcal{F}}(z)$ -conjugate to xz , another contradiction. We conclude that $\text{foc}(\mathcal{N}) = D$ is of index 2 in S . As $S/\text{foc}(\mathcal{N})$ is cyclic this shows that $\text{hnp}(\mathcal{N}) = \text{foc}(\mathcal{N}) = D$ is of index 2 as well by Lemma 1.11(c).

Let $\mathcal{M} = O^2(\mathcal{N})$, a saturated fusion system on D . Set $\mathcal{M}^+ = \mathcal{M}/\langle xz \rangle$, and let $\tau: \mathcal{M} \rightarrow \mathcal{M}^+$ denote the surjective morphism of fusion systems. Thus \mathcal{M}^+ is a saturated fusion system on $D^+ = D/Z(S)$, a product $D_1^+ \times D_2^+$ of dihedral groups each of order 2^k , and with $\mathcal{M}^+ = O^2(\mathcal{M}^+)$ by Lemma 1.14. Since $k \geq 3$, $\text{Aut}(D^+)$ is a 2-group by Lemma 1.4, and so it follows that $\mathcal{M}^+ = O^{2'}(\mathcal{M}^+)$ as well, by Proposition 1.13. The hypotheses of Theorem 1.15 are now satisfied and therefore $\mathcal{M}^+ \cong \mathcal{M}_1^+ \times \mathcal{M}_2^+$, by that theorem, for some pair \mathcal{M}_i^+ of saturated fusion systems on D_i^+ . Note then that the $\mathcal{M}_i^+ = O^2(\mathcal{M}_i^+)$ are determined as the unique perfect 2-fusion system on the dihedral group D_i^+ (Lemmas 1.14 and 1.25), i.e. as $\mathcal{F}_{D_i^+}(\mathcal{M}_i^+)$ with $\mathcal{M}_i^+ \cong L_2(q)$.

Let \mathcal{M}_i be the full preimage of \mathcal{M}_i^+ under τ ; \mathcal{M}_i is a saturated fusion system on D_i . Then $\mathcal{M}_i = O^2(\mathcal{M}_i)$ since $Z(S) = Z(\mathcal{M}_i) \leq [D_i, D_i]$, and $\mathcal{M}_i/Z(\mathcal{M}_i) = \mathcal{M}_i^+$ for each $i = 1, 2$. As there are no perfect fusion systems on a semidihedral group with nontrivial center by Lemma 1.25(b), each of the D_i is quaternion, and hence \mathcal{M}_i is the 2-fusion system of $SL_2(q)$ by (c) of the same lemma. Furthermore as f interchanges D_1 and D_2 , f interchanges \mathcal{M}_1 and \mathcal{M}_2 .

In particular, we conclude that D is a commuting product of quaternion groups of the same order, and $S \cong Q_{2^{k+1}} \wr C_2$. Thus S is not isomorphic to $SD_{2^{k+1}} \wr C_2$ as the latter has an involution with centralizer isomorphic to $C_2 \times SD_{2^{k+1}}$, whereas the former does not. By Lemma 6.18(i) and Lemma 6.23(i), h is an involution. \square

We now extract fusion information from the description of the centralizer of the central involution in Lemma 6.24 to show

Lemma 6.25. $h = 1$.

Proof. Suppose $h \neq 1$. Then the structure of S is that of Lemma 6.18 and $\mathcal{N} = C_{\mathcal{F}}(xz)$ is given by Lemma 6.24. Let $\mathcal{N}^+ = \mathcal{N}/\langle xz \rangle$ as before, and denote passage to the quotient by pluses. Recall $S = D_i \wr C_2$, with f a wreathing element and with each D_i quaternion and given as in Lemma 6.18(h). We claim

$$(6.26) \quad \text{every element of } J_0 f f^a \text{ is } \mathcal{N}\text{-conjugate to } x,$$

and once shown, this contradicts Lemma 6.13(a).

To see (6.26), note that $Z(S^+) = \langle x^+ \rangle$, and the image of each element of $J_0 f f^a = J_0 a^f a = J_0(xa)^f(xa)$ in S^+ is an involution which is not contained in either of the D_i^+ factors of the base subgroup of S^+ . Thus, by the structure of \mathcal{N}^+ , each such element has image in S^+ which is \mathcal{N}^+ conjugate to x^+ . Pulling back over the surjective morphism $\mathcal{N} \rightarrow \mathcal{N}^+$, it follows that each element of $J_0 f f^a$ is \mathcal{N} -conjugate into $Z(T) = \langle x, z \rangle$, and hence \mathcal{N} -conjugate to x . This finishes the proof of (6.26) and the lemma. \square

Lemma 6.27. \mathcal{F} is the fusion system of $L_4(q_1)$ for some $q_1 \equiv 3 \pmod{4}$ with $\nu_2(q_1 + 1) = k - 1$.

Proof. By Lemma 6.25, $S = J\langle a \rangle$ is isomorphic to $D_{2^k} \wr C_2$. By [Oli, Proposition 5.5(a)], either $k = 3$ and \mathcal{F} is the fusion system of A_{10} , or \mathcal{F} is the fusion system of $L_4(q_1)$ for $q_1 \equiv 3$

(mod 4) with $\nu_2(q_1 + 1) = k - 1$. In the case of A_{10} , x is a product of two transpositions with centralizer having a unique component \mathcal{K} isomorphic to the fusion system of $A_6 \cong L_2(9)$. But then $Q = C_T(\mathcal{K})$ is a four group, contrary to hypothesis. \square

Lemma 6.27 completes the identification of \mathcal{F} and the proof of Theorem 6.1.

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